Prime distances in colorings of the plane

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Theorem (Davies 2024)

In any coloring of the plane with finitely many **colors**, *there exist* $x, y \in \mathbb{R}^2$ *of the same color such that* $||x - y||$ *is* **odd**.

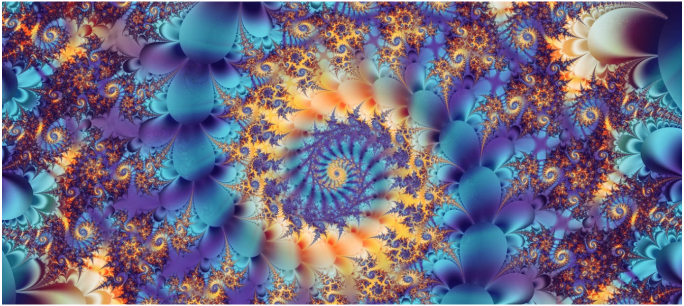

Figure by Andy Bantly

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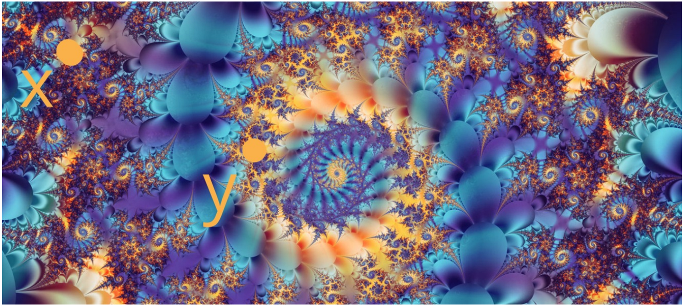

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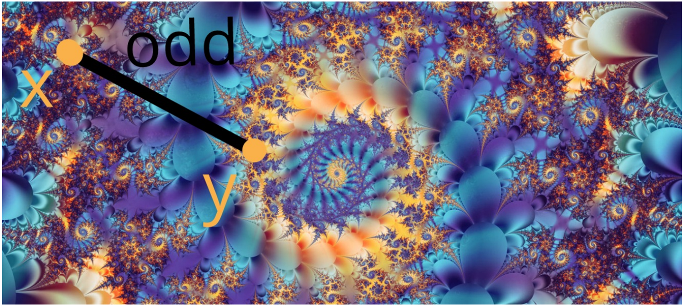

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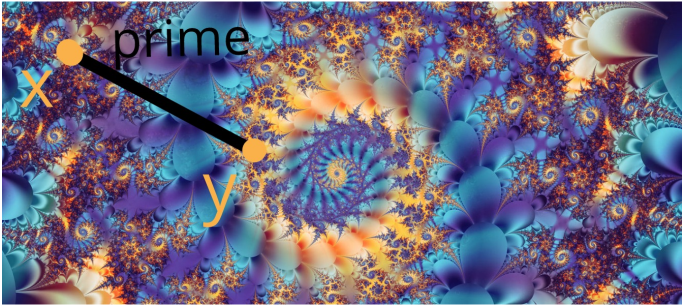
Theorem (Davies, M., Pilipczuk 2024+)

In any coloring of the plane with finitely many **colors***, there exist* $x, y \in \mathbb{R}^2$ *of the same color such that* $||x - y||$ *is* **prime***.*



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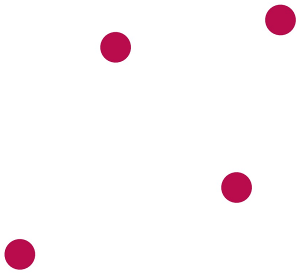

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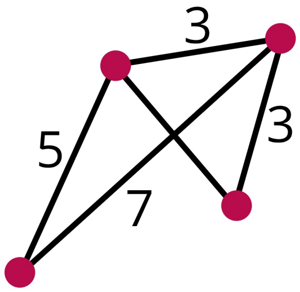
Let $\mathbf{P} \subseteq \mathbb{R}^2$ be finite. The **prime distance graph** has an edge between $x, y \in \mathbf{P}$ if $||x - y||$ is **prime**.



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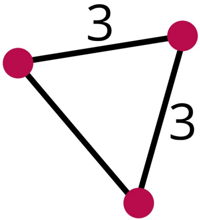

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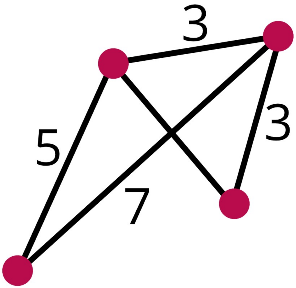

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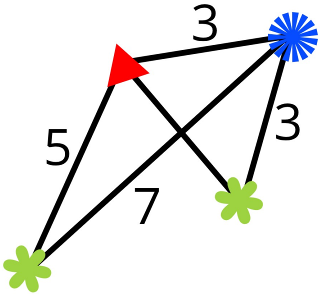
Let **P** $\subseteq \mathbb{R}^2$ be finite. The **prime distance graph** has an edge between $x, y \in$ **P** if $||x - y||$ is **prime**. **Theorem:** For each k, there exists such a graph of **chromatic number** $\geq k$.



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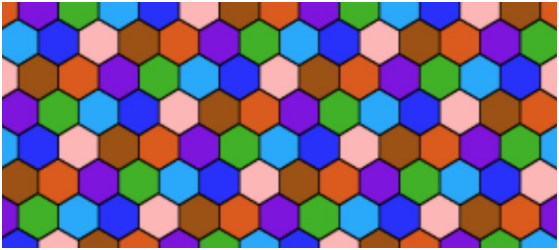

Figure by Daniel Ashlock

Theorem (Isbell; see Soifer 2008)

The plane can be colored with 7 **colors** *so that* **no** $x, y \in \mathbb{R}^2$ *of the same color have* $||x - y|| = 1$*.*

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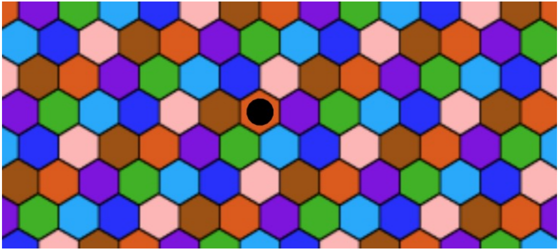

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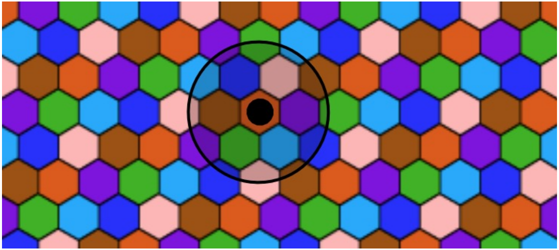

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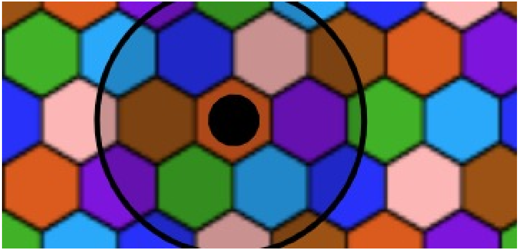

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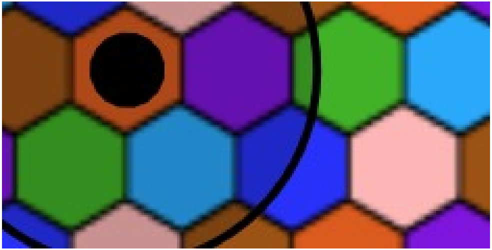

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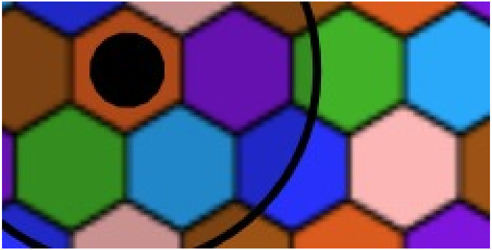
and ≥ 100



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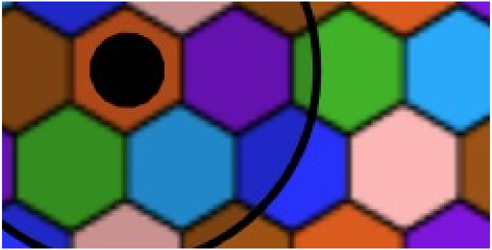

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Hadwiger–Nelson Problem. Aubrey de Grey: ≥ 5 colors needed.

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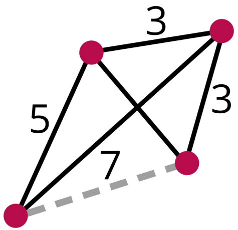

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There are no four points in $\mathbb{R}^2$ *whose pairwise distances are all odd.*

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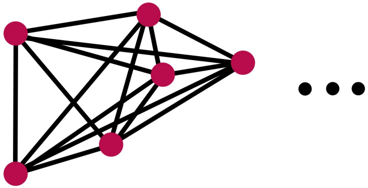

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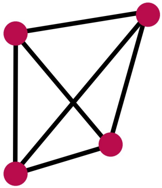

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If each color class is **measurable***, then there exists d_0 so that the* **"densest"** *color contains all real distances $d \geq d_0$.*

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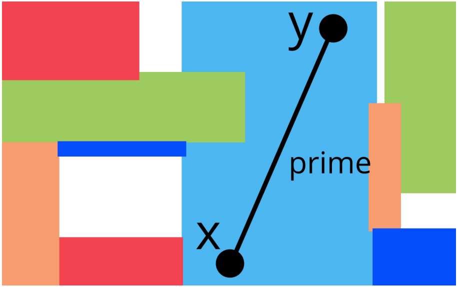

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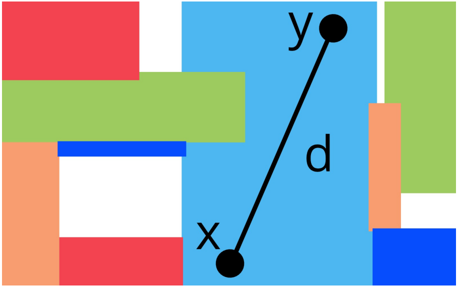

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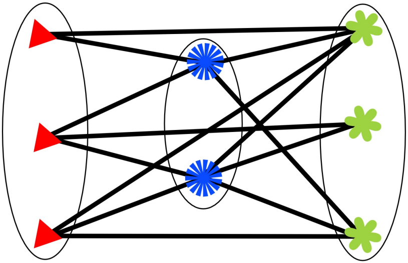

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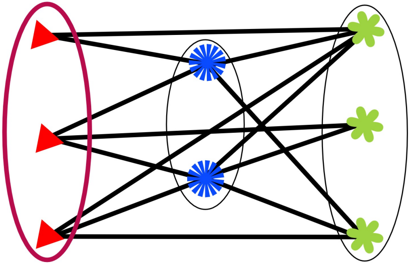

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$$\limsup_{\ell \to \infty} \frac{m(S_\ell \cap I)}{m(S_\ell)} > 0.$$

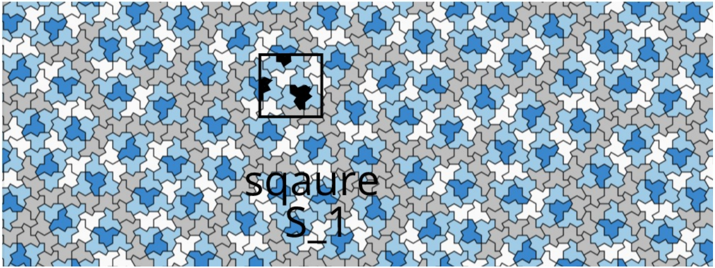

Figure by Smith, Myers, Kaplan, and Goodman-Strauss

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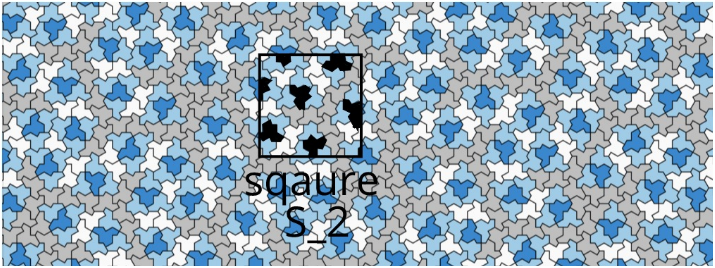

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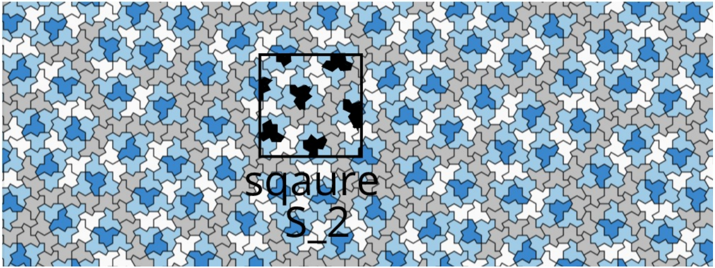

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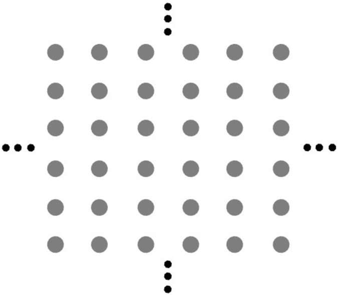
To define “density” in the **non-measurable** setting, we go to $\mathbb{Z}^2$.



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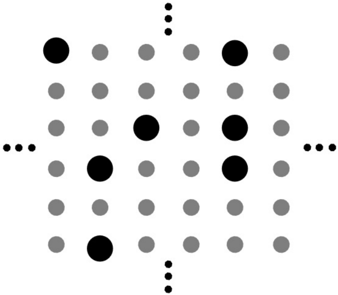

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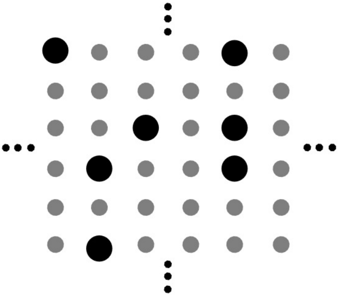

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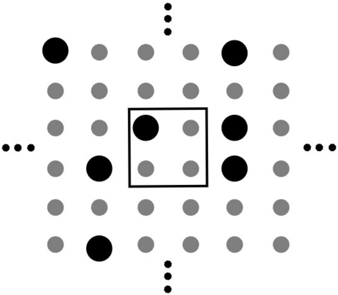

$$\frac{|S_1 \cap I|}{|S_1|} = \frac{1}{4}$$

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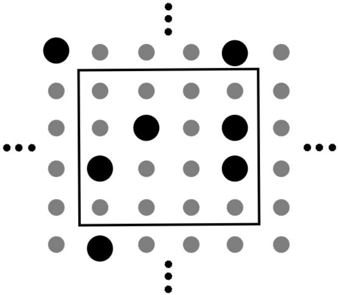

$$\frac{|S_2 \cap I|}{|S_2|} = \frac{4}{16}$$

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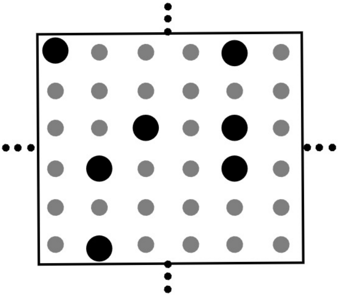

$$\frac{|S_3 \cap I|}{|S_3|} = \frac{7}{36}$$

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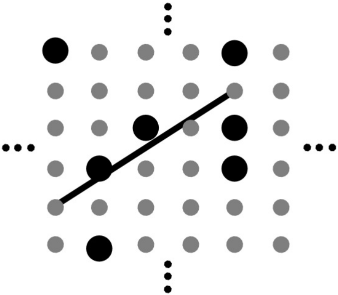

$$\sqrt{3^2 + 4^2} = 5$$

Put an edge between pairs whose distance is **prime**.

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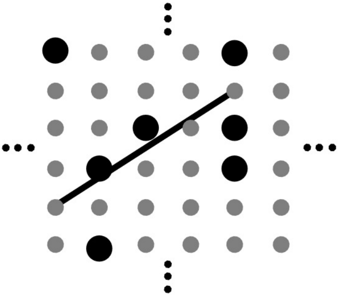

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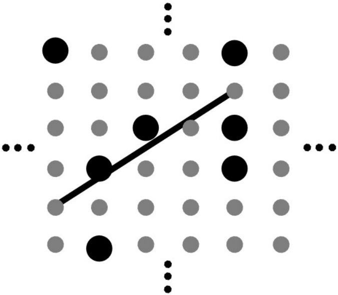

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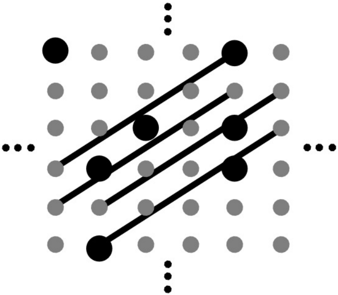

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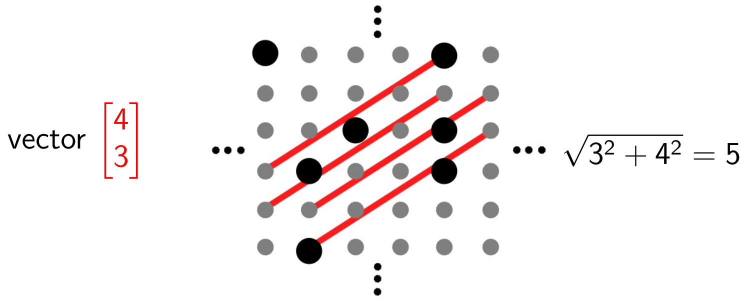

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Given a finite set X of integer vectors called the **generators**, the **Cayley graph** $G(\mathbb{Z}^2, X)$ has edges between v and $v \pm x$ for each $v \in \mathbb{Z}^2$ and $x \in X$.

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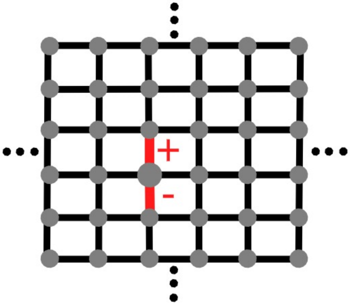

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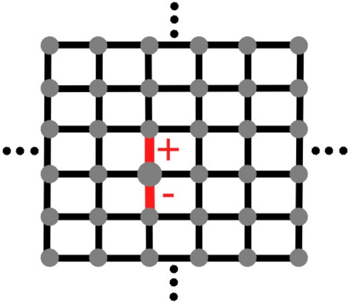

We define a particular X so that

- the vectors in X have different directions, and
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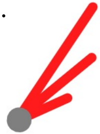

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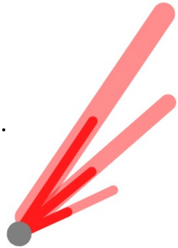

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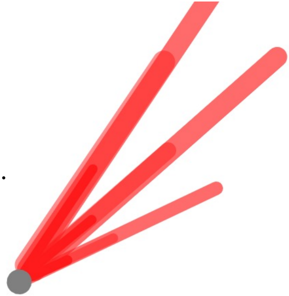

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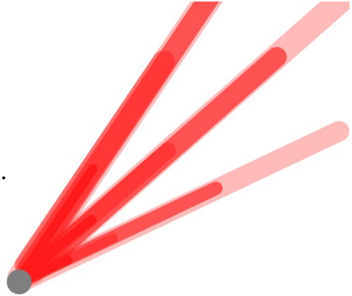

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For any k, there exists an X so that every **independent set** *of $G(\mathbb{Z}^2, \mathbb{P}X)$ has* **upper density** $< 1/k$.

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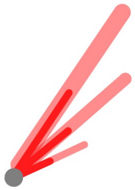

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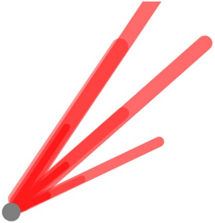

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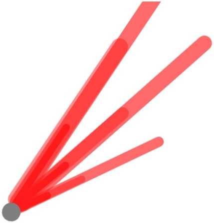

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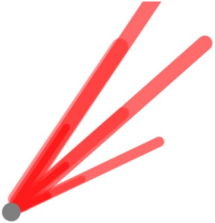

Theorem (Lovász 1979)

*Every independent set in a d-***regular** *graph has density $\leq -\lambda_{min}/(d - \lambda_{min})$.*

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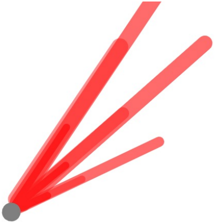

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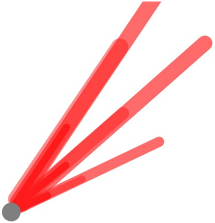

Theorem (Lovász 1979)

*Every independent set in a d-**regular** graph has density $\leq -\lambda_{min}/d$, and this holds in the* **edge-weighted setting**.

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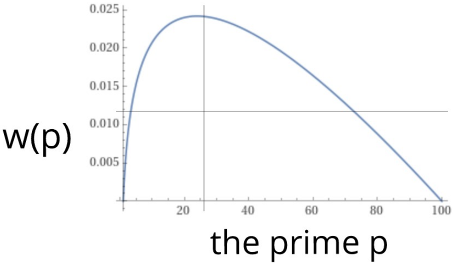
$$w(p) = \frac{1}{N}\left(1 - \frac{p}{N}\right)\ln(p)$$



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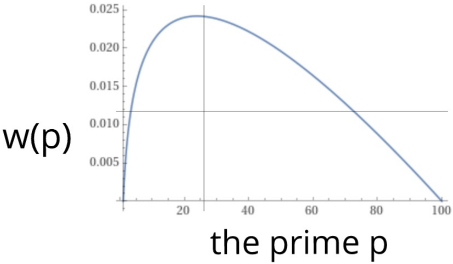

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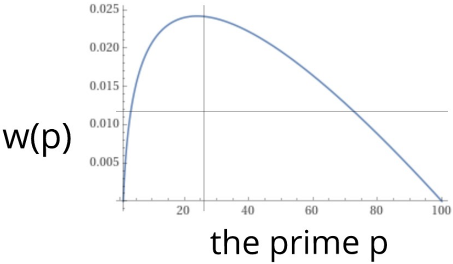

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Theorem (Davies 2024): In this edge-weighted setting, we essentially have $\lambda_{min} = \inf_{u \in \mathbb{R}^2} \sum_{px} w(p) \cos(2\pi(u \cdot x))$.

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Tools:

- A more precise PNT due to Poussin (1899).
- Vinogradov's estimates for certain exponential sums over primes, used to show that every sufficiently large odd integer is a sum of three primes (1937).

Theorem (Davies, M., Pilipczuk 2024+)

In any coloring of the plane with finitely many colors, there exist $x, y \in \mathbb{R}^2$ of the same color such that $||x - y|| = f(\mathbb{Z})$.

For any non-constant integer polynomial f with leading coefficient > 0, i.e. $f(x) = x^2 + 3$, $f(\mathbb{Z}) = \{3, 4, 7, 12, \ldots\}$.

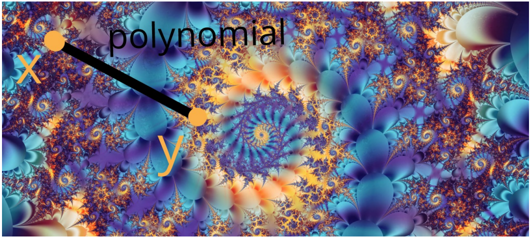

Figure by Andy Bantly

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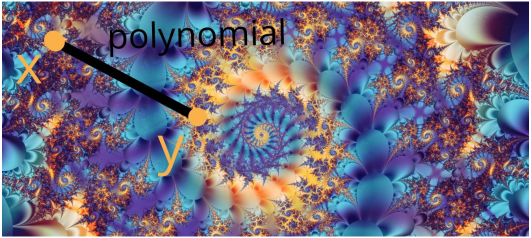

Question

Is there any **infinite** *set $D \subseteq \mathbb{Z}$ so that the plane* **can be** *colored with finitely many colors so as to avoid distances in D?*

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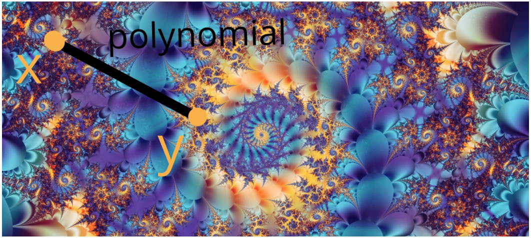

Question (Soifer 2010)

What if $D = \{2^n : n \in \mathbb{N}\}$, or $D = \{n! : n \in \mathbb{N}\}$?

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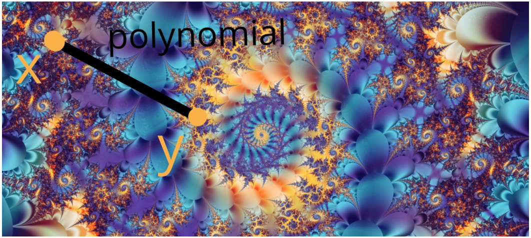

Conjecture (Bukh 2008)

For any **algebraically independent** *$D \subseteq \mathbb{R}$, the plane* **can be** *colored with finitely many colors so as to avoid distances in D.*

Thank you!