Colorings of the plane cannot avoid prime distances

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Theorem (Davies)

In any coloring of the plane with finitely many colors, there exist monochromatic $x, y \in \mathbb{R}^2$ such that $||x - y||$ is **odd**.

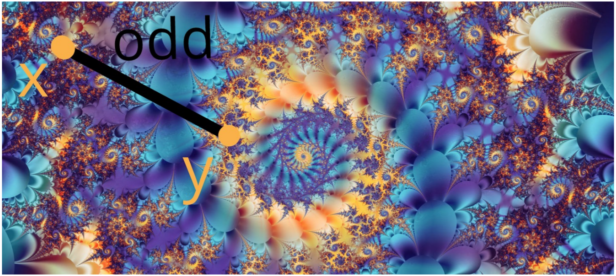

Figure by Andy Bantly

Theorem (Davies, M., Pilipczuk 2023)

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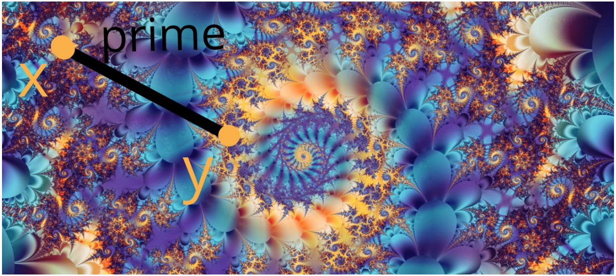

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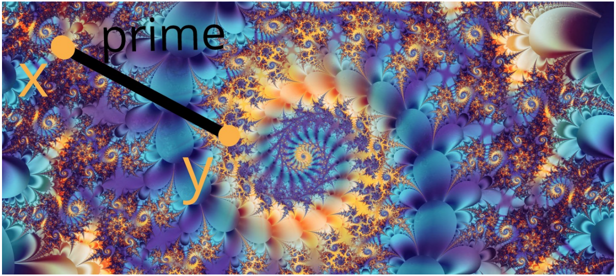

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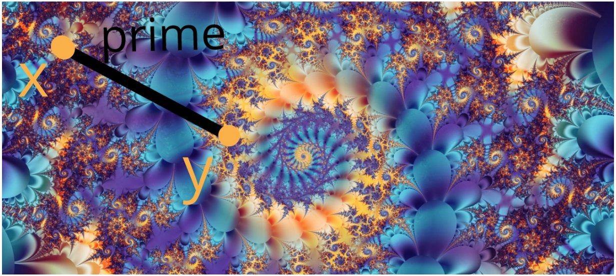

Theorem (Fürstenberg, Katznelson, Weiss 1990)

This is true if each color class is **measurable**. *In fact, the* **"densest"** *color contains all distances* $d \geq d_0$.

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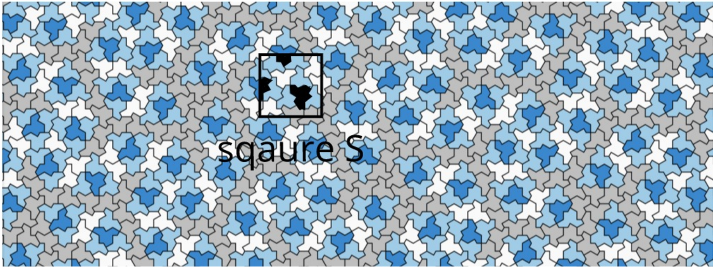
$$\limsup_{\ell \to \infty} \frac{m(S \cap I)}{m(S)} > 0.$$



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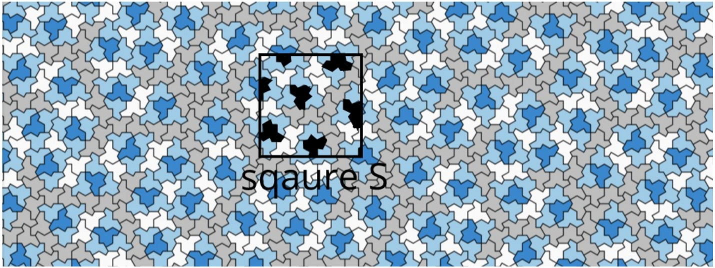

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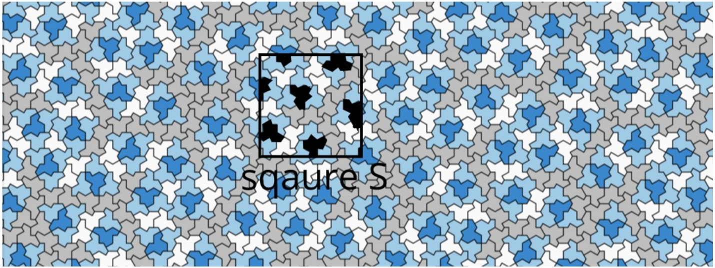

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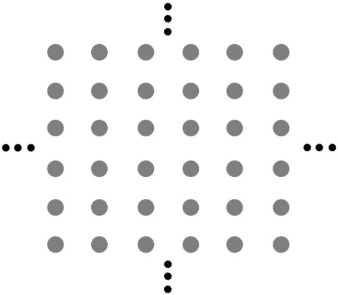
In any **measurable** coloring of $\mathbb{R}^2$ with finitely many colors, some color class has positive upper density.

To define “density” in the **non-measurable** setting, we consider $I \subseteq \mathbb{Z}^2$.



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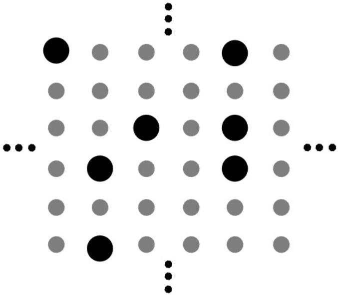

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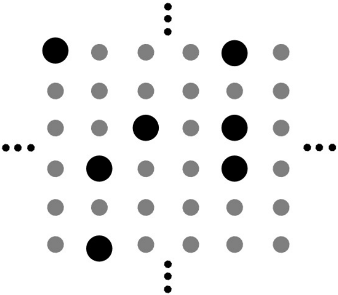

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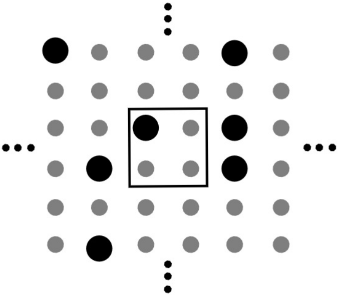

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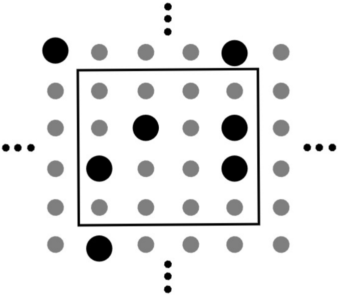

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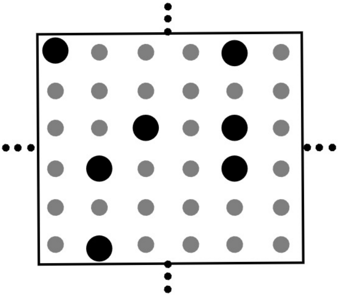

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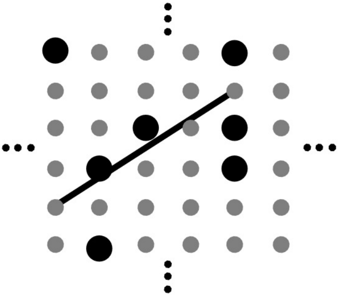

$$\sqrt{3^2 + 4^2} = 5$$

Put an edge between pairs whose distance is **prime**.

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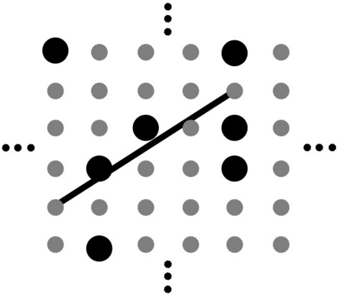

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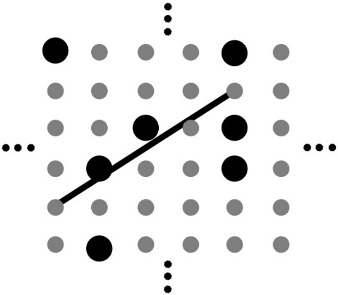

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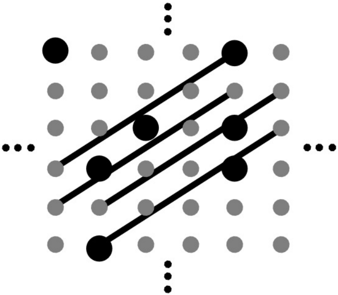

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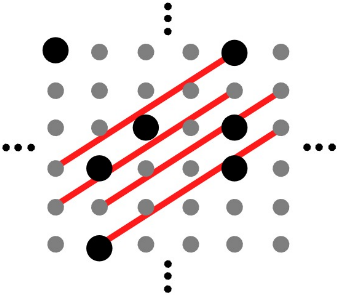
vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$



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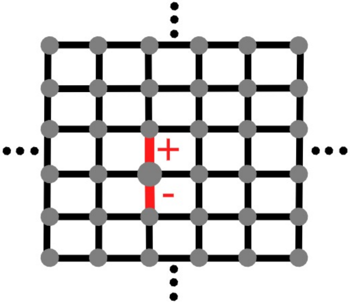
Given a finite set X of integer vectors, the **Cayley graph** $G(\mathbb{Z}^2, X)$ has edges between v and $v \pm x$ for each $x \in X$.

$$X = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$



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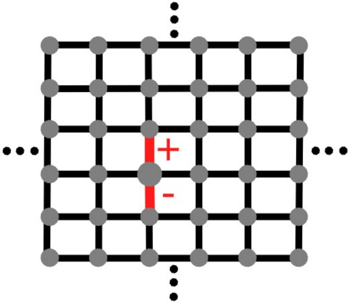

We define a particular X so that

- the vectors in X have different directions, and
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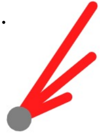

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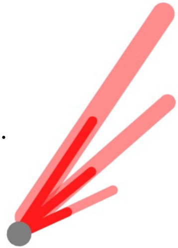

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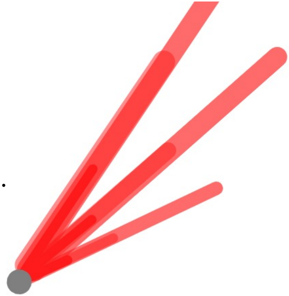

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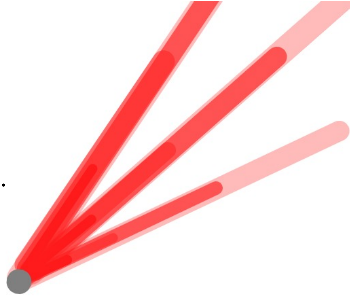

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Additional property: For each $u \in \mathbb{R}^2$, few $x \in X$ have $u \cdot x$ being “close to” a non-zero rational with small denominator.

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Theorem (Davies 2022; inspired by the Lovász theta bound)

It suffices to find $w : \mathbb{P}X \to \mathbb{R}_{\geq 0}$ so that $w(\mathbb{P}X) = 1$ and $-\inf_{u \in \mathbb{R}^2} \sum_{x \in \mathbb{P}X} w(x) \cos(2\pi(u \cdot x)) < \epsilon$.

We really want a weight function $w : \mathbb{P} \to \mathbb{R}_{\geq 0}$ so that

$$-inf_{\alpha \in \mathbb{R}} \sum_{p \in \mathbb{P}} w(p) \cos(2\pi\alpha) < \epsilon w(\mathbb{P}),$$

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Then $\lim_{N\to\infty} w_N(\mathbb{P}) = \frac{1}{2}$ since the average prime between 1 and N is $\sim N/2$, and the prime number theorem says

$$\lim_{N\to\infty} \frac{1}{N} \sum_{p \leq N} \log p = 1.$$

Question

Is there any infinite subset $D \subseteq \mathbb{Z}$ so that the plane **can be** *colored with finitely many colors so as to avoid distances in D?*

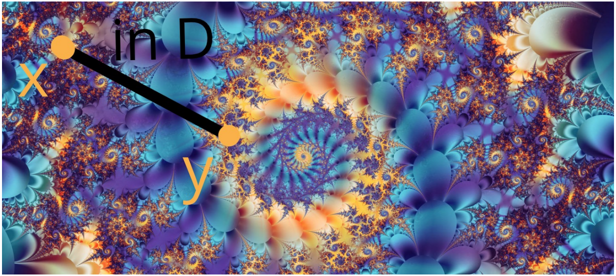

Figure by Andy Bantly

Thank you!