Colorings, graphs, and geometry

Rose McCarty

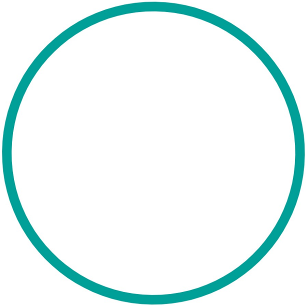
Schools of Math and CS

Georgia Tech

April 18th, 2024

Joint with James Davies and more...

Consider the unit circle $\mathbf{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.



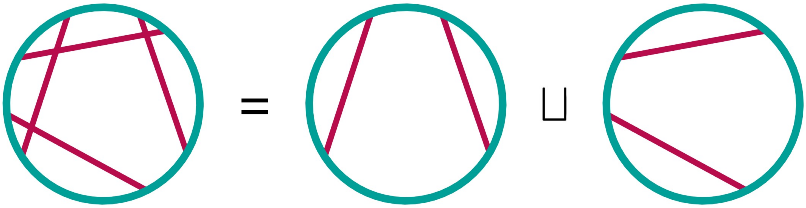
Consider the unit circle $\mathbf{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A **chord** is a line segment between two points on $\mathbf{C}$.

Consider the unit circle **C** $= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A **chord** is a line segment between two points on **C**.

Consider the unit circle $\mathbf{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A **chord** is a line segment between two points on $\mathbf{C}$.

Consider the unit circle **C** $= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A **chord** is a line segment between two points on **C**. Fix a finite collection $\mathcal{R}$ of chords.

Consider the unit circle **C** $= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A **chord** is a line segment between two points on **C**. Fix a finite collection $\mathcal{R}$ of chords. Can we partition $\mathcal{R}$ into $\leqslant t$ non-crossing parts?



Consider the unit circle **C** $= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A **chord** is a line segment between two points on **C**. Fix a finite collection $\mathcal{R}$ of chords. Can we partition $\mathcal{R}$ into $\leqslant t$ non-crossing parts?

One obstruction is $t + 1$ pairwise intersecting chords in $\mathcal{R}$.

Consider the unit circle $\mathbf{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A **chord** is a line segment between two points on $\mathbf{C}$. Fix a finite collection $\mathcal{R}$ of chords. Can we partition $\mathcal{R}$ into $\leqslant t$ non-crossing parts?

One obstruction is $t + 1$ pairwise intersecting chords in $\mathcal{R}$.

Theorem (Gyárfás 1985)

If $\mathcal{R}$ does not contain $t + 1$ pairwise intersecting chords, then it can be partitioned into at most $2^{2t} t^2$ non-intersecting parts.

Consider the unit circle **C** $= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A **chord** is a line segment between two points on **C**. Fix a finite collection $\mathcal{R}$ of chords. Can we partition $\mathcal{R}$ into $\leqslant t$ non-crossing parts?

One obstruction is $t + 1$ pairwise intersecting chords in $\mathcal{R}$.

Theorem (Kostochka and Kratochvíl 1997)

If $\mathcal{R}$ does not contain $t + 1$ pairwise intersecting chords, then it can be partitioned into at most $50 \cdot 2^t$ non-intersecting parts.

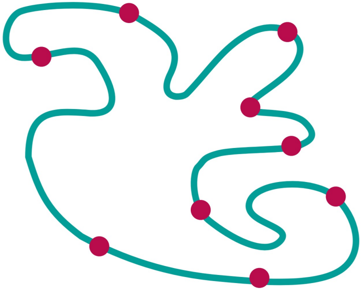
Consider the unit circle $\mathbf{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. A **chord** is a line segment between two points on $\mathbf{C}$. Fix a finite collection $\mathcal{R}$ of chords. Can we partition $\mathcal{R}$ into $\leqslant t$ non-crossing parts?

One obstruction is $t + 1$ pairwise intersecting chords in $\mathcal{R}$.

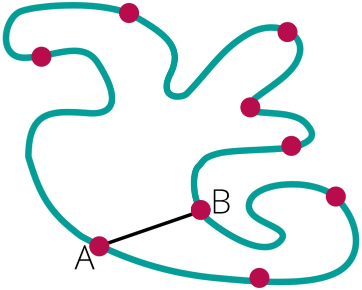
Theorem (Davies and McCarty 2021)

If $\mathcal{R}$ does not contain $t + 1$ pairwise intersecting chords, then it can be partitioned into at most $7t^2$ non-intersecting parts.

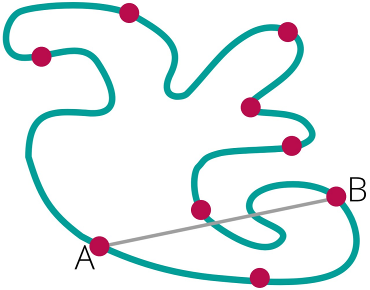
Consider a Jordan curve $\mathcal{J}$ and a finite set of points $\mathbf{P} \subset \mathcal{J}$.



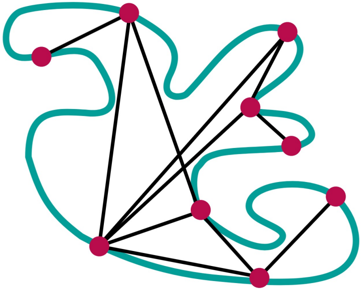
Consider a Jordan curve $\mathcal{J}$ and a finite set of points $\mathbf{P} \subset \mathcal{J}$. Two points in $\mathbf{P}$ are **visible** if the line segment between them is inside of $\mathcal{J}$.



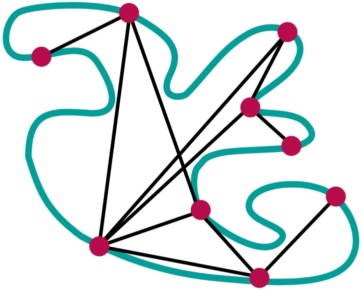
Consider a Jordan curve $\mathcal{J}$ and a finite set of points $\mathbf{P} \subset \mathcal{J}$. Two points in $\mathbf{P}$ are **visible** if the line segment between them is inside of $\mathcal{J}$.



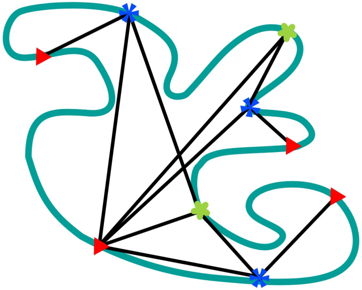
Consider a Jordan curve $\mathcal{J}$ and a finite set of points $\mathbf{P} \subset \mathcal{J}$. Two points in $\mathbf{P}$ are **visible** if the line segment between them is inside of $\mathcal{J}$.



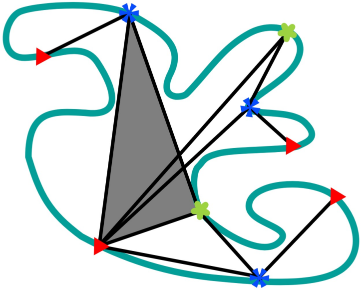
Consider a Jordan curve $\mathcal{J}$ and a finite set of points $\mathbf{P} \subset \mathcal{J}$. Two points in $\mathbf{P}$ are **visible** if the line segment between them is inside of $\mathcal{J}$. When can we partition $\mathbf{P}$ into $\leqslant t$ invisible parts?



Consider a Jordan curve $\mathcal{J}$ and a finite set of points $\mathbf{P} \subset \mathcal{J}$. Two points in $\mathbf{P}$ are **visible** if the line segment between them is inside of $\mathcal{J}$. When can we partition $\mathbf{P}$ into $\leqslant t$ invisible parts?

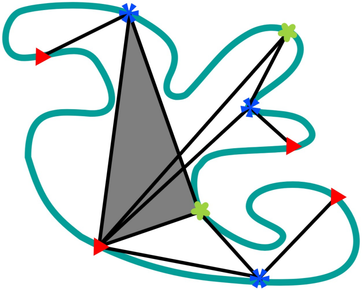


Consider a Jordan curve $\mathcal{J}$ and a finite set of points $\mathbf{P} \subset \mathcal{J}$. Two points in $\mathbf{P}$ are **visible** if the line segment between them is inside of $\mathcal{J}$. When can we partition $\mathbf{P}$ into $\leqslant t$ invisible parts?



One obstruction is $t + 1$ pairwise visible points in $\mathbf{P}$.

Consider a Jordan curve $\mathcal{J}$ and a finite set of points $\mathbf{P} \subset \mathcal{J}$. Two points in $\mathbf{P}$ are **visible** if the line segment between them is inside of $\mathcal{J}$. When can we partition $\mathbf{P}$ into $\leqslant t$ invisible parts?



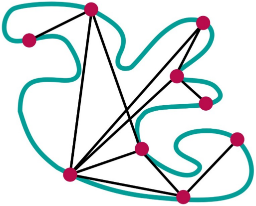
One obstruction is $t + 1$ pairwise visible points in $\mathbf{P}$.

Theorem (Davies, Krawczyk, McCarty, and Walczak 2021)

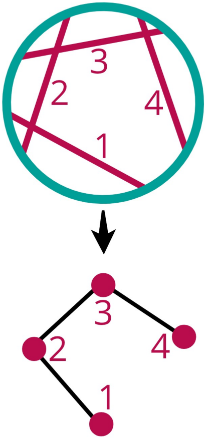
If $\mathbf{P}$ *does not contain* $t + 1$ *pairwise visible points, then it can be partitioned into at most* 4^t *invisible parts.*

Babia Góra, border of Slovakia and Poland, 2019

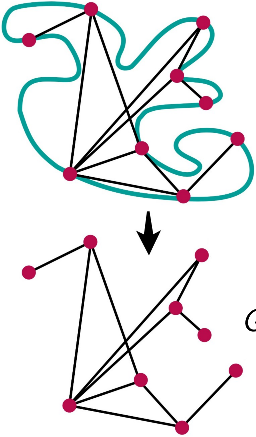
A general formulation



A general formulation



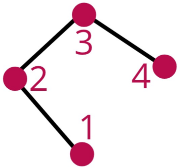
circle graph



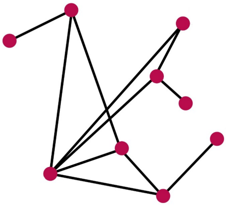
$G = (\mathbf{V}, E)$

curve visibility graph

A general formulation



circle graph

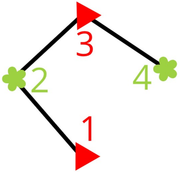


$G = (\mathbf{V}, E)$

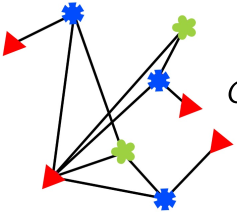
curve visibility graph

A general formulation

The **chromatic number** $\chi(G)$ is the minimum number of **colors** needed to assign adjacent vertices in **V** different colors.



circle graph



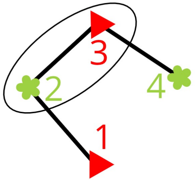
$G = (\mathbf{V}, E)$

curve visibility graph

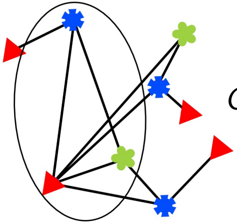
A general formulation

The **chromatic number** $\chi(G)$ is the minimum number of **colors** needed to assign adjacent vertices in **V** different colors.

The **clique number** $\omega(G)$ is the maximum size of a set of pairwise adjacent vertices in **V**.



circle graph



$G = (\mathbf{V}, E)$

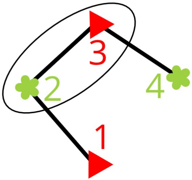
curve visibility graph

A general formulation

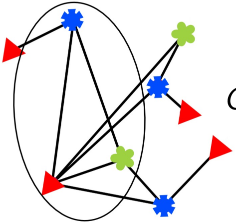
The **chromatic number** $\chi(G)$ is the minimum number of colors needed to assign adjacent vertices in **V** different colors.

The **clique number** $\omega(G)$ is the maximum size of a set of pairwise adjacent vertices in **V**.

$$\omega \leqslant \chi$$



circle graph



$G = (\mathbf{V}, E)$

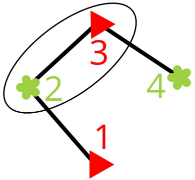
curve visibility graph

A general formulation

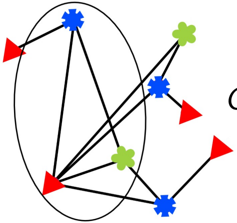
The **chromatic number** $\chi(G)$ is the minimum number of **colors** needed to assign adjacent vertices in **V** different colors.

The **clique number** $\omega(G)$ is the maximum size of a set of pairwise adjacent vertices in **V**.

$$\omega \leqslant \chi \leqslant f(\omega)$$



circle graph



$G = (\mathbf{V}, E)$

curve visibility graph

A general formulation

The **chromatic number** $\chi(G)$ is the minimum number of **colors** needed to assign adjacent vertices in **V** different colors.

The **clique number** $\omega(G)$ is the maximum size of a set of pairwise adjacent vertices in **V**.

$$\omega \leqslant \chi \leqslant f(\omega)$$

A class of graphs is χ-**bounded** if such a function exists, and f is called a χ-**bounding function**.

A general formulation

The **chromatic number** $\chi(G)$ is the minimum number of colors needed to assign adjacent vertices in **V** different colors.

The **clique number** $\omega(G)$ is the maximum size of a set of pairwise adjacent vertices in **V**.

$$\omega \leqslant \chi \leqslant f(\omega)$$

A class of graphs is χ-**bounded** if such a function exists, and f is called a χ-**bounding function**.

Theorem (Erdös 1959)

The class of all graphs is not χ-bounded.

A general formulation

The **chromatic number** $\chi(G)$ is the minimum number of **colors** needed to assign adjacent vertices in **V** different colors.

The **clique number** $\omega(G)$ is the maximum size of a set of pairwise adjacent vertices in **V**.

$$\omega \leqslant \chi \leqslant f(\omega)$$

A class of graphs is χ-**bounded** if such a function exists, and f is called a χ-**bounding function**.

Theorem (Erdös 1959)

The class of all graphs is not χ-bounded.

Vizing's Theorem, the Strong Perfect Graph Theorem,
Gyárfás–Sumner Conjecture, ...

How quickly can an optimal χ-bounding function grow?

$$\chi \leqslant \omega$$

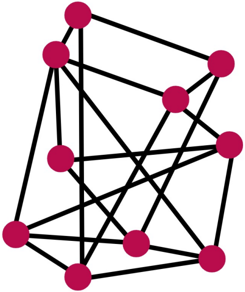
$$\chi \leqslant \omega^3$$

$$\chi \leqslant 2^\omega$$

$$\chi \leqslant \omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}}$$

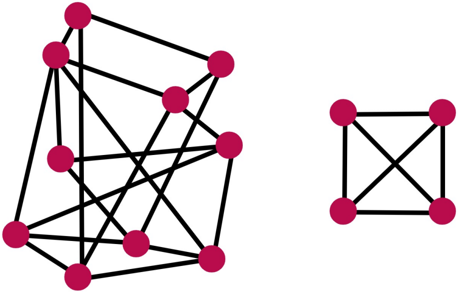
$$\dots$$

How quickly can an optimal χ-bounding function grow?



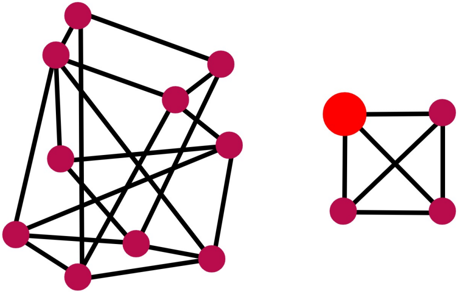
$$\chi(G) = k$$
$$\omega(G) = 2$$

How quickly can an optimal χ-bounding function grow?



$$\chi(G) = k$$
$$\omega(G) = 4$$

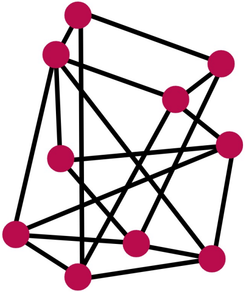
How quickly can an optimal χ-bounding function grow?



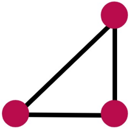
$$\chi(G) = k$$

$$\omega(G) = 4$$

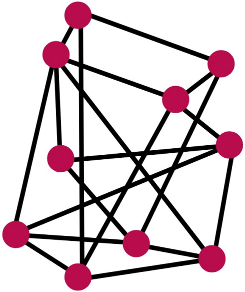
How quickly can an optimal χ-bounding function grow?



$\chi(G) = k$
$\omega(G) = 3$

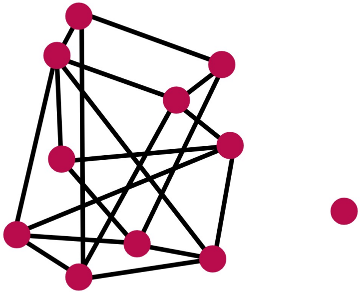


How quickly can an optimal χ-bounding function grow?



$$\chi(G) = k$$
$$\omega(G) = 2$$

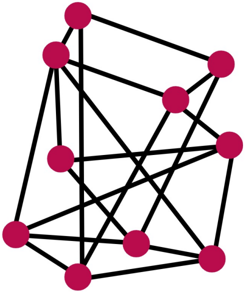
How quickly can an optimal χ-bounding function grow?



$\chi(G) = k$

$\omega(G) = 2$

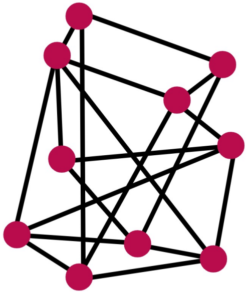
How quickly can an optimal χ-bounding function grow?



$\chi(G) = k$

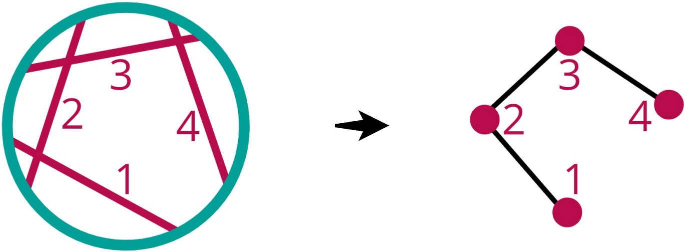
$\omega(G) = 2$

How quickly can an optimal χ-bounding function grow?



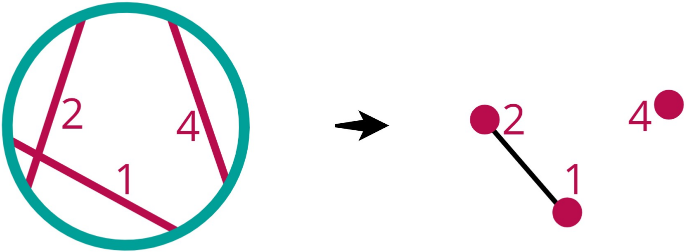
We only consider classes that are closed under vertex-deletion.

How quickly can an optimal χ-bounding function grow?



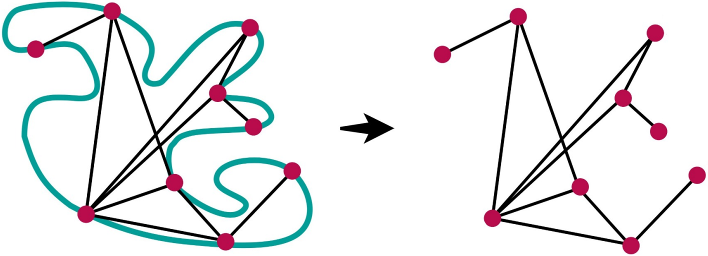
We only consider classes that are closed under vertex-deletion.

How quickly can an optimal χ-bounding function grow?



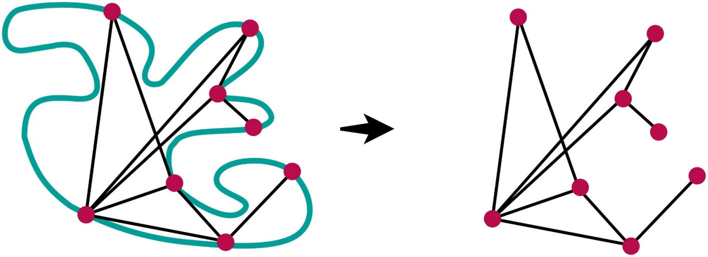
We only consider classes that are closed under vertex-deletion.

How quickly can an optimal χ-bounding function grow?



We only consider classes that are closed under vertex-deletion.

How quickly can an optimal χ-bounding function grow?



We only consider classes that are closed under vertex-deletion.

How quickly can an optimal
χ-bounding function grow?

Esperet's Conjecture

There is always a polynomial χ-bounding function.

We only consider classes that are
closed under vertex-deletion.

How quickly can an optimal χ-bounding function grow?

Esperet's Conjecture

There is always a polynomial χ-bounding function.

i.e. if $\chi \leqslant \omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}}$ then $\chi \leqslant \omega^d$ too!

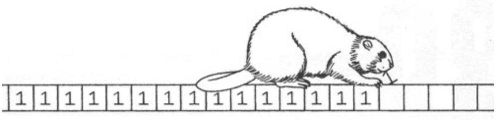


Figure from The New Turing Omnibus, Dewdney

We only consider classes that are closed under vertex-deletion.

How quickly can an optimal χ-bounding function grow?

Esperet's Conjecture

There is always a polynomial χ-bounding function.

i.e. if $\chi \leqslant \omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}}$ then $\chi \leqslant \omega^d$ too!

Theorem (Briański, Davies, and Walczak 2023)

Actually, optimal χ-bounding functions can grow arbitrarily quickly.

We only consider classes that are closed under vertex-deletion.

Theorem (Erdös 1959)

The class of all graphs is not χ*-bounded.*

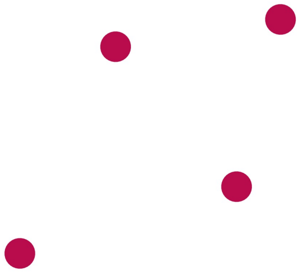
Theorem (Davies, McCarty, and Pilipczuk 2024+)

The class of all **prime distance graphs** *is not χ-bounded.*

Theorem (Davies, McCarty, and Pilipczuk 2024+)

The class of all **prime distance graphs** *is* *not* *χ-bounded.*

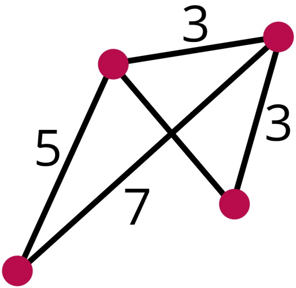
Let $\mathbf{P} \subseteq \mathbb{R}^2$.



Theorem (Davies, McCarty, and Pilipczuk 2024+)

The class of all **prime distance graphs** *is not χ-bounded.*

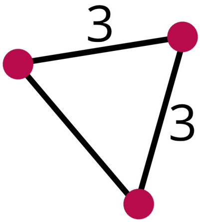
Let **P** $\subseteq \mathbb{R}^2$. Put an edge between $x, y \in$ **P** if $||x - y||$ is **prime**.



Theorem (Davies, McCarty, and Pilipczuk 2024+)

The class of all **prime distance graphs** *is* *not* *χ-bounded.*

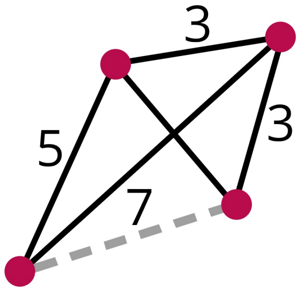
Let **P** $\subseteq \mathbb{R}^2$. Put an edge between $x, y \in$ **P** if $||x - y||$ is **prime**.



Theorem (Davies, McCarty, and Pilipczuk 2024+)

The class of all **prime distance graphs** *is* *not* χ*-bounded.*

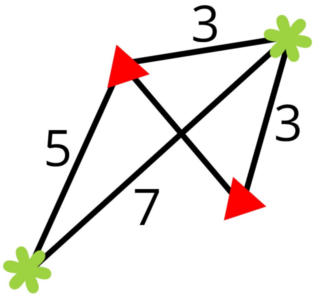
Let $\mathbf{P} \subseteq \mathbb{R}^2$. Put an edge between $x, y \in \mathbf{P}$ if $||x - y||$ is **prime**.
There is no 4-vertex clique (Graham, Rothschild, and Straus 1974).



Theorem (Davies, McCarty, and Pilipczuk 2024+)

The class of all **prime distance graphs** *is not* χ*-bounded.*

Let $\mathbf{P} \subseteq \mathbb{R}^2$. Put an edge between $x, y \in \mathbf{P}$ if $||x - y||$ is **prime**.
There is no 4-vertex clique (Graham, Rothschild, and Straus 1974).
Our Theorem: For all k, there exists such a graph with $\chi \geqslant k$.



Theorem (Davies, McCarty, and Pilipczuk 2024+)

In any coloring of the plane with finitely many colors, there exist $x, y \in \mathbb{R}^2$ of the same color such that $||x - y||$ is **prime***.*

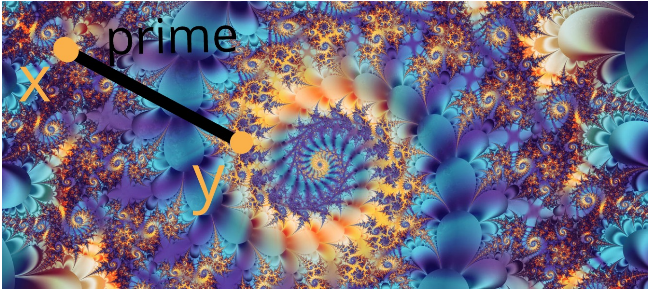
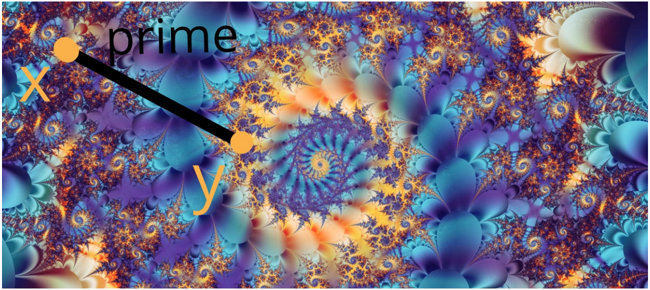


Figure by Andy Bantly

Theorem (Davies, McCarty, and Pilipczuk 2024+)

In any coloring of the plane with finitely many colors, there exist $x, y \in \mathbb{R}^2$ of the same color such that $||x - y||$ is **prime***.*

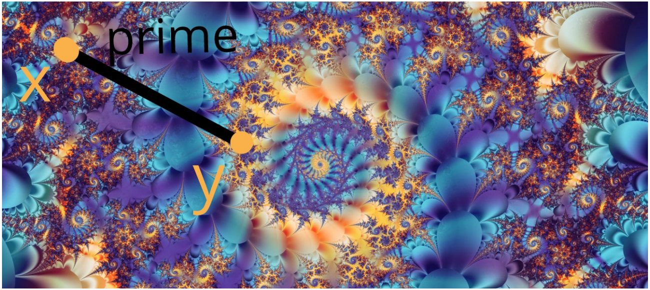


Theorem (Fürstenberg, Katznelson, Weiss 1990)

This is true if each color class is **measurable***.*

Theorem (Davies, McCarty, and Pilipczuk 2024+)

In any coloring of the plane with finitely many colors, there exist $x, y \in \mathbb{R}^2$ of the same color such that $||x - y||$ is ***prime****.*



Theorem (Fürstenberg, Katznelson, Weiss 1990)

This is true if each color class is ***measurable****. In fact, the* ***"densest"*** *color contains all sufficiently large distances in $\mathbb{R}$.*

A measurable set $I \subseteq \mathbb{R}^2$ has **positive upper density** if

$$\limsup_{|S| \to \infty} \frac{m(S \cap I)}{m(S)} > 0.$$

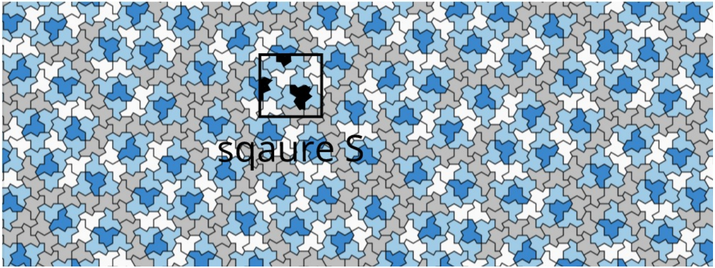


Figure by Smith, Myers, Kaplan, and Goodman-Strauss

A measurable set $I \subseteq \mathbb{R}^2$ has **positive upper density** if

$$\limsup_{|S| \to \infty} \frac{m(S \cap I)}{m(S)} > 0.$$

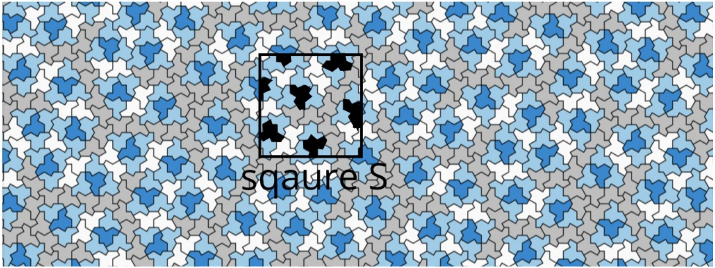


Figure by Smith, Myers, Kaplan, and Goodman-Strauss

A measurable set $I \subseteq \mathbb{R}^2$ has **positive upper density** if

$$\limsup_{|S|\to\infty} \frac{m(S \cap I)}{m(S)} > 0.$$

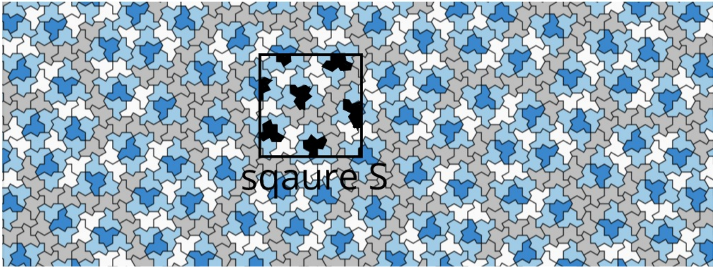
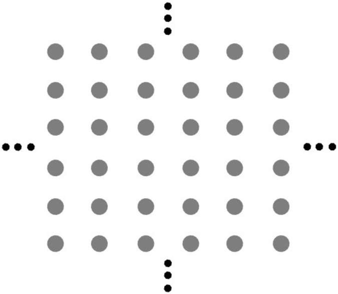


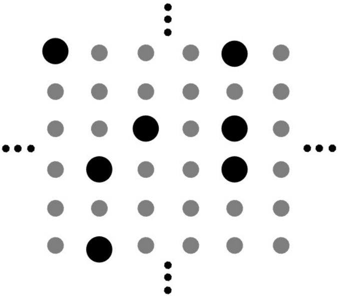
Figure by Smith, Myers, Kaplan, and Goodman-Strauss

Fact: In any **measurable** coloring of $\mathbb{R}^2$ with finitely many colors, there exists a color of positive upper density.

To define “density” in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$.

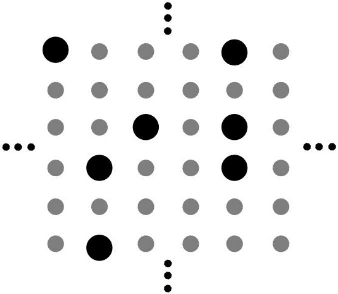


To define “density” in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$.



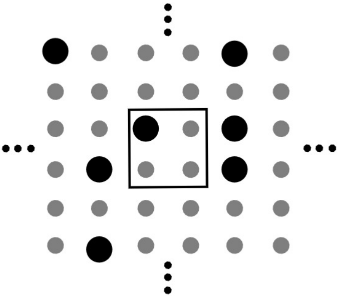
To define "density" in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$.
The set I has **positive upper density** if

$$\limsup_{|S|\to\infty} \frac{|S \cap I|}{|S|} > 0.$$



To define "density" in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$.
The set I has **positive upper density** if

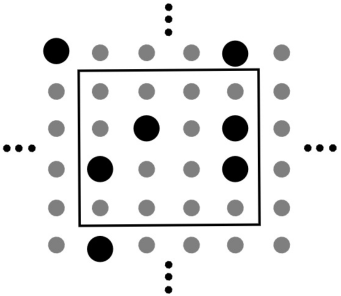
$$\limsup_{|S|\to\infty} \frac{|S \cap I|}{|S|} > 0.$$



$$\frac{|S \cap I|}{|S|} = \frac{1}{4}$$

To define "density" in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$.
The set I has **positive upper density** if

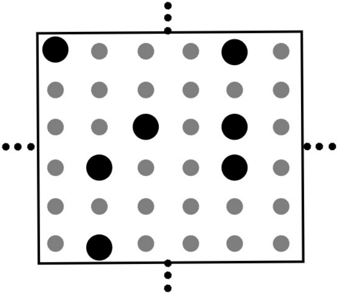
$$\limsup_{|S|\to\infty} \frac{|S \cap I|}{|S|} > 0.$$



$$\frac{|S \cap I|}{|S|} = \frac{4}{16}$$

To define “density” in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$.
The set I has **positive upper density** if

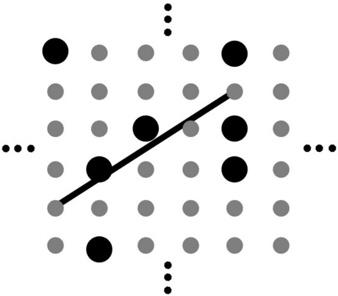
$$\limsup_{|S|\to\infty} \frac{|S \cap I|}{|S|} > 0.$$



$$\frac{|S \cap I|}{|S|} = \frac{7}{36}$$

To define "density" in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$.
The set I has **positive upper density** if

$$\limsup_{|S|\to\infty} \frac{|S \cap I|}{|S|} > 0.$$

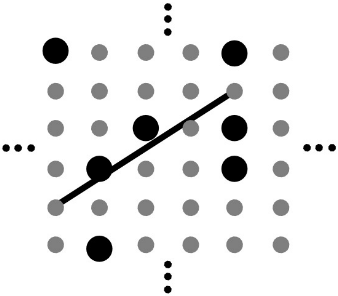


$$\sqrt{4^2 + 3^2} = 5$$

Put an edge between pairs whose distance is **prime**.

To define "density" in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$. The set I has **positive upper density** if

$$\limsup_{|S|\to\infty} \frac{|S \cap I|}{|S|} > 0.$$

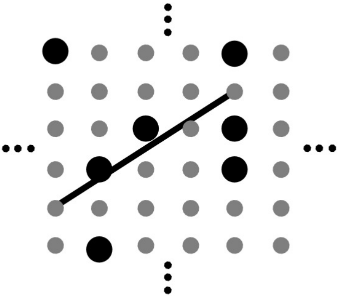


$$\sqrt{4^2 + 3^2} = 5$$

Put an edge between pairs whose distance is **prime**. This graph actually has **chromatic number** 2 :(.

To define "density" in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$. The set I has **positive upper density** if

$$\limsup_{|S| \to \infty} \frac{|S \cap I|}{|S|} > 0.$$

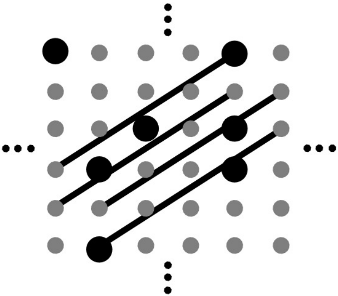


$\sqrt{4^2 + 3^2} = 5$

Put an edge between pairs whose distance is **prime**. This graph actually has **chromatic number** 2 :(. So we define a different graph on $\mathbb{Z}^2$ which **embeds** in the prime distance graph on $\mathbb{R}^2$.

To define "density" in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$. The set I has **positive upper density** if

$$\limsup_{|S|\to\infty} \frac{|S \cap I|}{|S|} > 0.$$



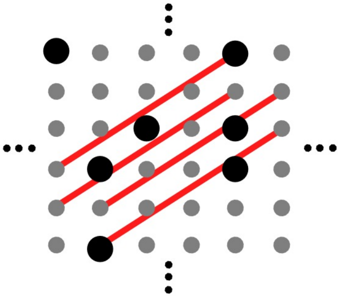
$\sqrt{4^2 + 3^2} = 5$

Put an edge between pairs whose distance is **prime**. This graph actually has **chromatic number** 2 :(. So we define a different graph on $\mathbb{Z}^2$ which **embeds** in the prime distance graph on $\mathbb{R}^2$.

To define "density" in the **non-measurable** setting, let $I \subseteq \mathbb{Z}^2$. The set I has **positive upper density** if

$$\limsup_{|S|\to\infty} \frac{|S \cap I|}{|S|} > 0.$$

vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

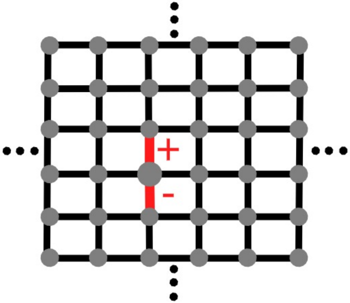


$\sqrt{4^2 + 3^2} = 5$

Put an edge between pairs whose distance is **prime**. This graph actually has **chromatic number** 2 :(. So we define a different graph on $\mathbb{Z}^2$ which **embeds** in the prime distance graph on $\mathbb{R}^2$.

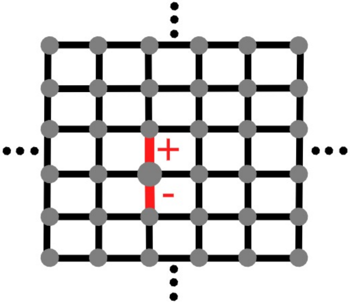
Given a finite set X of integer vectors, the **Cayley graph** $G(\mathbb{Z}^2, X)$ has edges between v and $v \pm x$ for each $x \in X$.

$$X = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$



Given a finite set X of integer vectors, the **Cayley graph** $G(\mathbb{Z}^2, X)$ has edges between v and $v \pm x$ for each $x \in X$.

$$X = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$



We define a particular X so that

- the vectors in X have different directions, and
- $G(\mathbb{Z}^2, \mathbb{P}X)$ embeds in the prime distance graph on $\mathbb{R}^2$.

Given a finite set X of integer vectors, the **Cayley graph** $G(\mathbb{Z}^2, X)$ has edges between v and $v \pm x$ for each $x \in X$.

$X = ???$

We define a particular X so that

- the vectors in X have different directions, and
- $G(\mathbb{Z}^2, \mathbb{P}X)$ embeds in the prime distance graph on $\mathbb{R}^2$.

Given a finite set X of integer vectors, the **Cayley graph** $G(\mathbb{Z}^2, X)$ has edges between v and $v \pm x$ for each $x \in X$.

$$X = ???$$

$$\mathbb{P}X = 2X \cup 3X \cup 5X \cup 7X \cup \dots$$

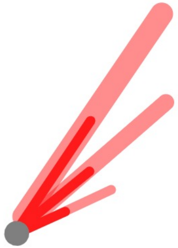
We define a particular X so that

- the vectors in X have different directions, and
- $G(\mathbb{Z}^2, \mathbb{P}X)$ embeds in the prime distance graph on $\mathbb{R}^2$.

Given a finite set X of integer vectors, the **Cayley graph** $G(\mathbb{Z}^2, X)$ has edges between v and $v \pm x$ for each $x \in X$.

$$X = ???$$

$$\mathbb{P}X = 2X \cup 3X \cup 5X \cup 7X \cup \ldots$$



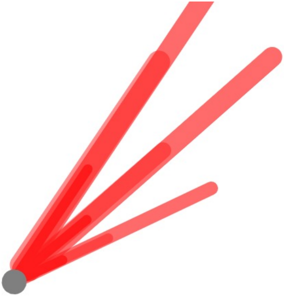
We define a particular X so that

- the vectors in X have different directions, and
- $G(\mathbb{Z}^2, \mathbb{P}X)$ embeds in the prime distance graph on $\mathbb{R}^2$.

Given a finite set X of integer vectors, the **Cayley graph** $G(\mathbb{Z}^2, X)$ has edges between v and $v \pm x$ for each $x \in X$.

$$X = ???$$

$$\mathbb{P}X = 2X \cup 3X \cup 5X \cup 7X \cup \dots$$



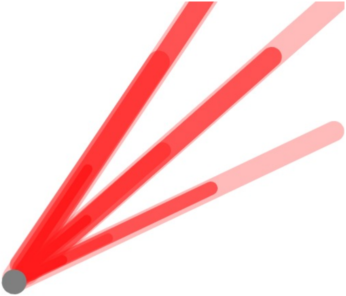
We define a particular X so that

- the vectors in X have different directions, and
- $G(\mathbb{Z}^2, \mathbb{P}X)$ embeds in the prime distance graph on $\mathbb{R}^2$.

Given a finite set X of integer vectors, the **Cayley graph** $G(\mathbb{Z}^2, X)$ has edges between v and $v \pm x$ for each $x \in X$.

$$X = ???$$

$$\mathbb{P}X = 2X \cup 3X \cup 5X \cup 7X \cup \dots$$



We define a particular X so that

- the vectors in X have different directions, and
- $G(\mathbb{Z}^2, \mathbb{P}X)$ embeds in the prime distance graph on $\mathbb{R}^2$.

Theorem (Davies, McCarty, and Pilipczuk 2024+)

For any k, we can choose X so that every **edgeless set** *of vertices in $G(\mathbb{Z}^2, \mathbb{P}X)$ has* **upper density** $< 1/k$.

Theorem (Davies, McCarty, and Pilipczuk 2024+)

For any k, we can choose X so that every **edgeless set** *of vertices in $G(\mathbb{Z}^2, \mathbb{P}X)$ has* **upper density** $< 1/k$.

It follows that $G(\mathbb{Z}^2, \mathbb{P}X)$ has **chromatic number** $> k$.

Theorem (Davies, McCarty, and Pilipczuk 2024+)

For any k, we can choose X so that every **edgeless set** *of vertices in $G(\mathbb{Z}^2, \mathbb{P}X)$ has* **upper density** $< 1/k$.

It follows that $G(\mathbb{Z}^2, \mathbb{P}X)$ has **chromatic number** $> k$.

Additional property of X: For each $u \in \mathbb{R}^2$, few $x \in X$ have $u \cdot x$ being "close to" a non-zero rational with small denominator.

Theorem (Davies, McCarty, and Pilipczuk 2024+)

For any k, we can choose X so that every **edgeless set** *of vertices in $G(\mathbb{Z}^2, \mathbb{P}X)$ has* **upper density** $< 1/k$.

Few $x \in X$ have $||u \cdot x - \frac{a}{b}|| < \epsilon$ for coprime a, b with $|b|$ small.

It follows that $G(\mathbb{Z}^2, \mathbb{P}X)$ has **chromatic number** $> k$.

Additional property of X: For each $u \in \mathbb{R}^2$, few $x \in X$ have $u \cdot x$ being "close to" a non-zero rational with small denominator.

Theorem (Davies, McCarty, and Pilipczuk 2024+)

For any k, we can choose X so that every **edgeless set** *of vertices in $G(\mathbb{Z}^2, \mathbb{P}X)$ has* **upper density** $< 1/k$.

Few $x \in X$ have $||u \cdot x - \frac{a}{b}|| < \epsilon$ for coprime a, b with $|b|$ small.

Theorem (Davies 2024; inspired by the Lovász theta bound)

It suffices to find a **weight function** $w : \mathbb{P}X \to \mathbb{R}_{\geqslant 0}$ *so that* $w(\mathbb{P}X) = 1$ *and* $-\inf_{u \in \mathbb{R}^2} \sum_{x \in \mathbb{P}X} w(x) \cos(2\pi(u \cdot x)) < \epsilon$.

Theorem (Davies, McCarty, and Pilipczuk 2024+)

For any k, we can choose X so that every **edgeless set** *of vertices in $G(\mathbb{Z}^2, \mathbb{P}X)$ has* **upper density** $< 1/k$.

The **prime number theorem** says that $\sum_{p \leqslant N} \frac{\log p}{N} \to 1$ as $N \to \infty$.

Few $x \in X$ have $||u \cdot x - \frac{a}{b}|| < \epsilon$ for coprime a, b with $|b|$ small.

Theorem (Davies 2024; inspired by the Lovász theta bound)

It suffices to find a **weight function** $w : \mathbb{P}X \to \mathbb{R}_{\geqslant 0}$ *so that* $w(\mathbb{P}X) = 1$ *and* $-\inf_{u \in \mathbb{R}^2} \sum_{x \in \mathbb{P}X} w(x) \cos(2\pi(u \cdot x)) < \epsilon$.

Question (Davies, McCarty, and Pilipczuk 2024+)

Is there **any infinite** *set $D \subseteq \mathbb{Z}$ so that the plane* **can be** *colored with finitely many colors so as to avoid distances in D?*

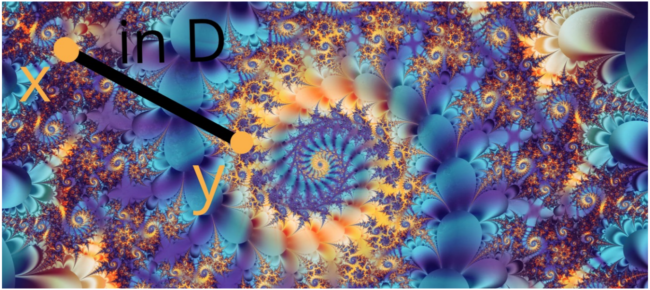
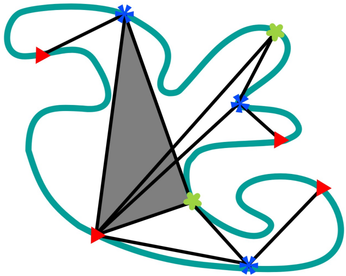


Figure by Andy Bantly

Question (Davies, McCarty, and Pilipczuk 2024+)

Is there **any infinite** *set $D \subseteq \mathbb{Z}$ so that the plane* **can be** *colored with finitely many colors so as to avoid distances in D?*



Question (Davies, Krawczyk, McCarty, and Walczak 2021)

Is there a **polynomial** *p so that* **curve visibility graphs** *with clique number ω have chromatic number $\leqslant p(\omega)$?*

Thank you!

Puerto Rico, 2023