Neighborhood complexity and matroids

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Georgia Tech

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LSU Combinatorics Seminar

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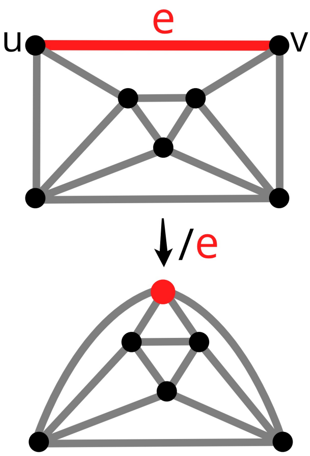

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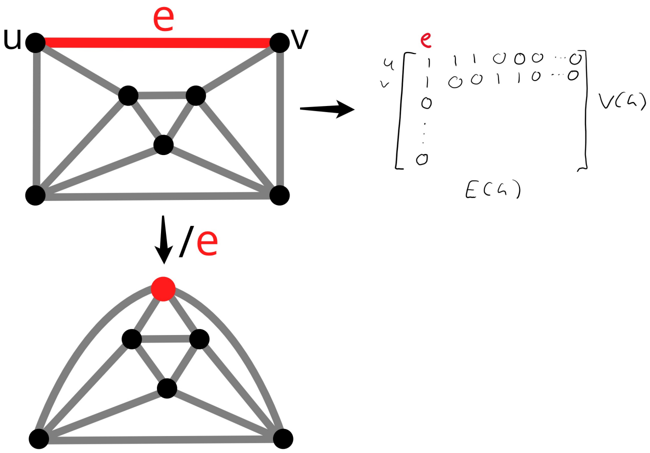

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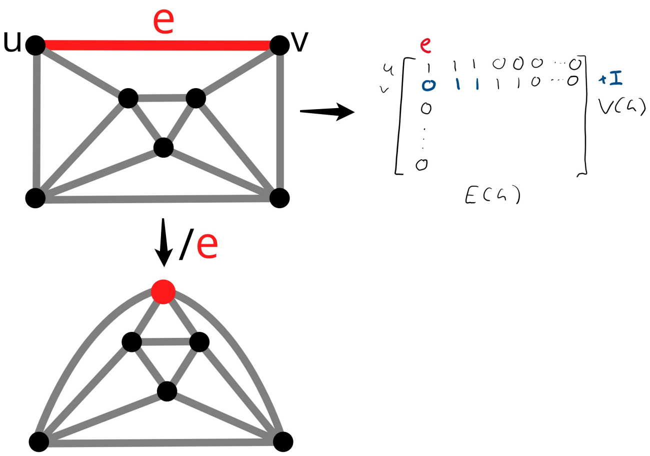

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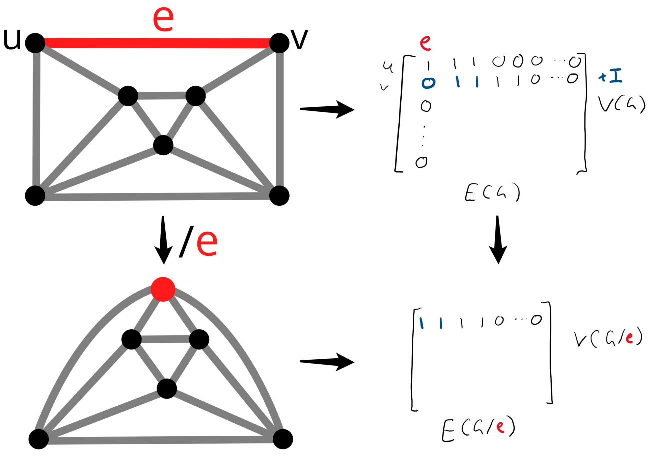

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$U_{2,4}$

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$$\begin{array}{c} e \\ \end{array}$$

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$$E(M)$$

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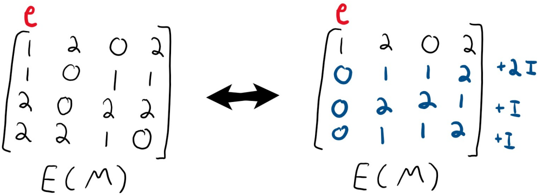

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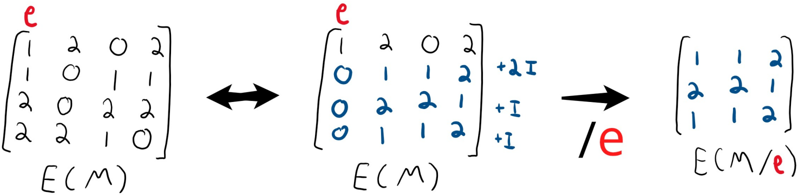

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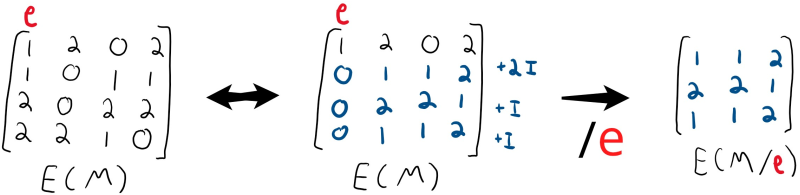

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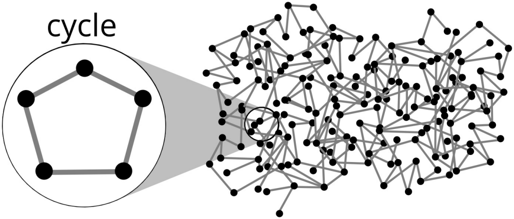

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$$\underbrace{2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{\text{minimize}} = 0$$

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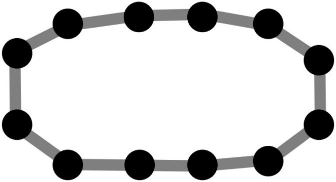

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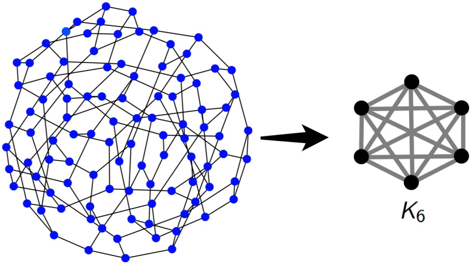

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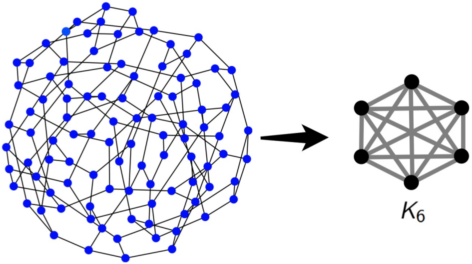

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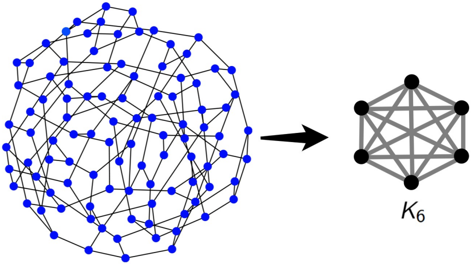

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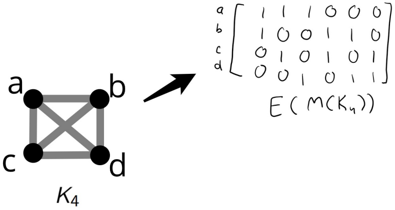

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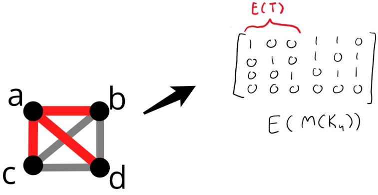

K_4

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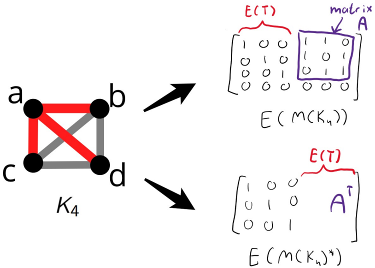

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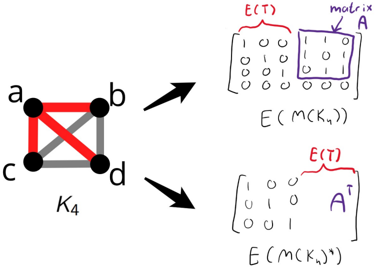

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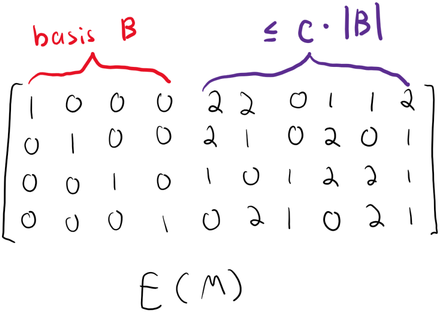

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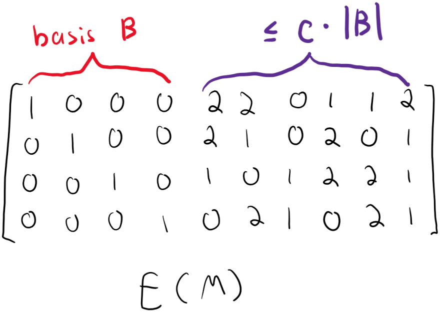

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$$\underset{E(M)}{\left[\overbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}}^{B}\;\begin{matrix}1&1&0&1\\1&0&1&1\\0&1&1&1\end{matrix}\right]}$$

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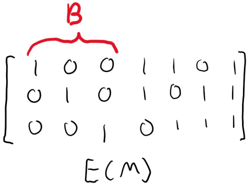

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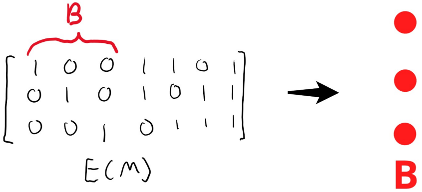

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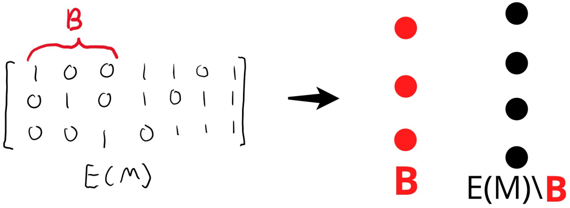

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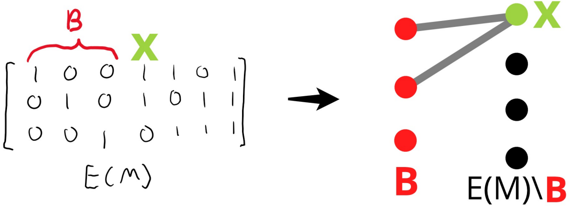

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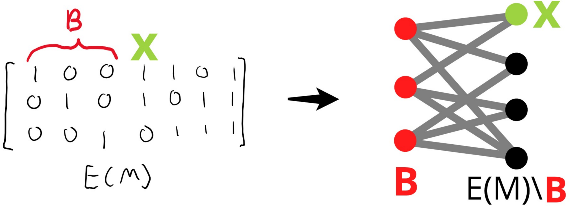

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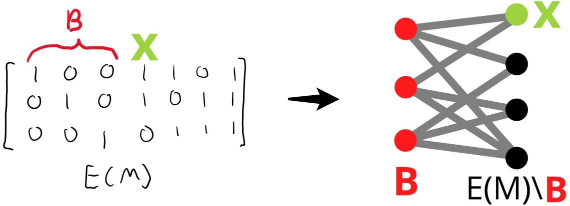

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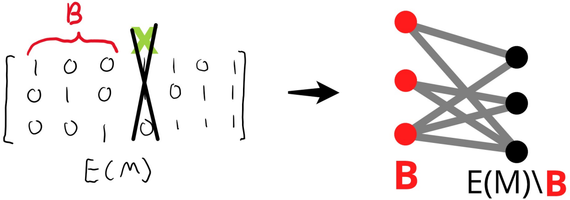

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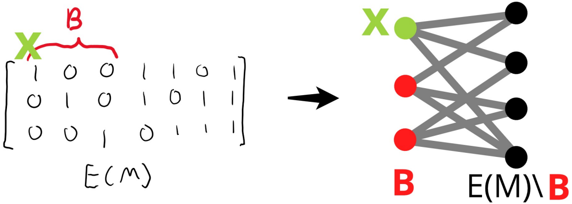

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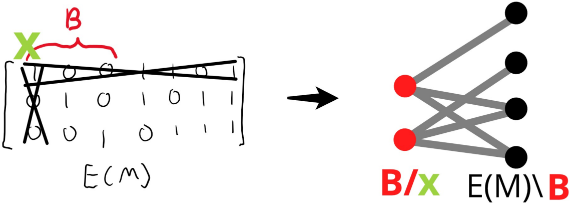

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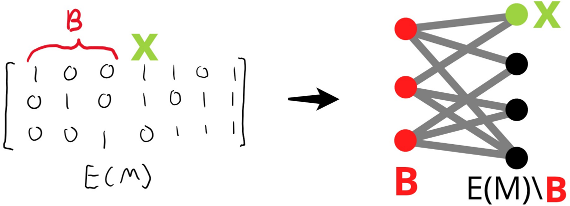

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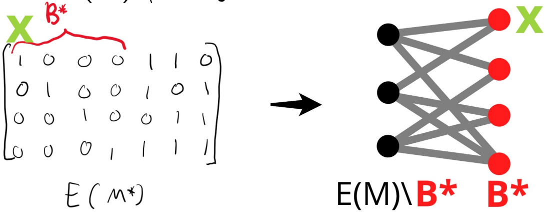

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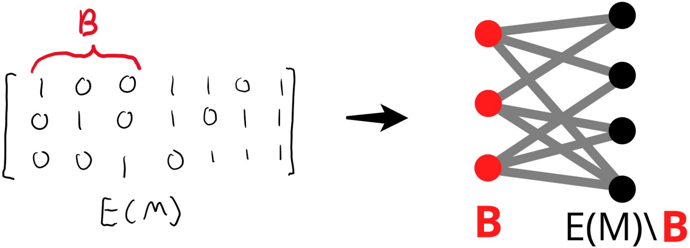

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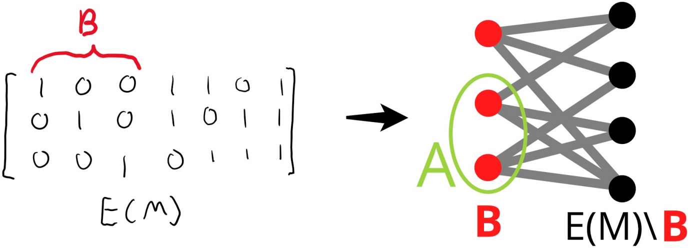

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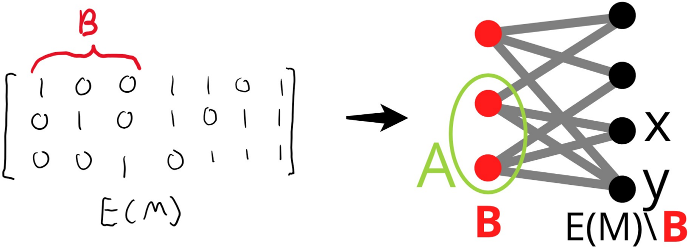

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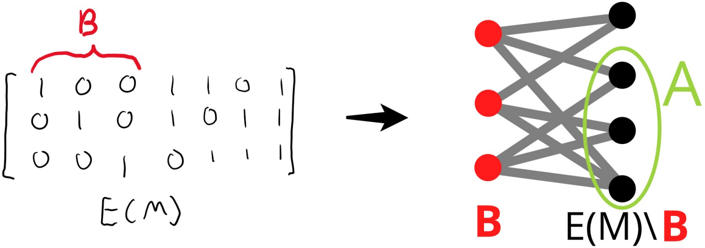

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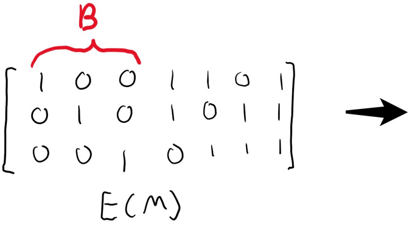

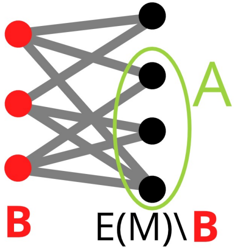

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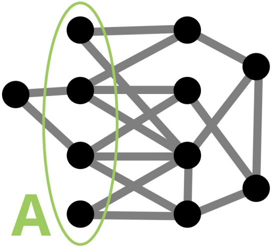

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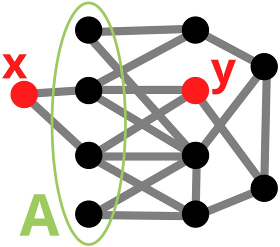

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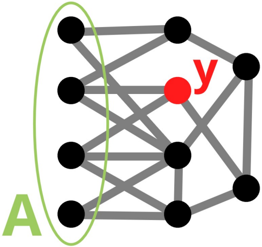

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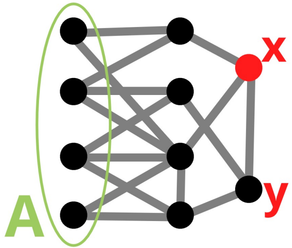

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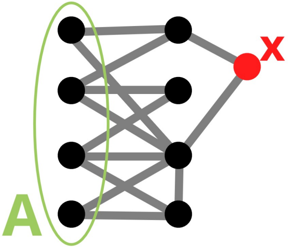

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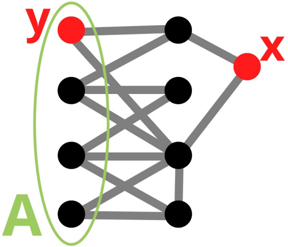

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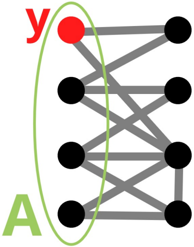

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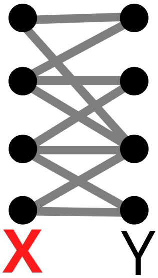
For any bipartite graph $G = (\mathbf{X}, Y)$ *with no twins in* **X** *and* $\eta(m) \le c \cdot m \ \forall m$,



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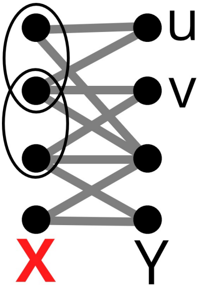

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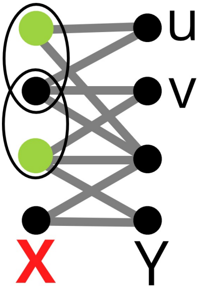

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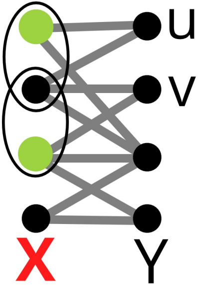
Such vertices are called **near-twins**.



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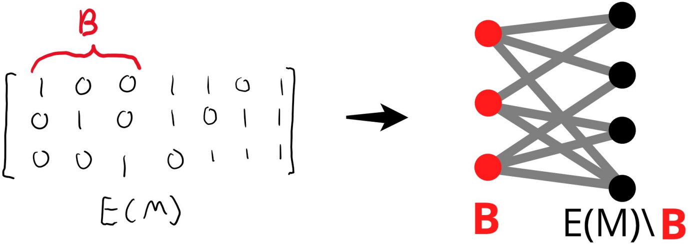

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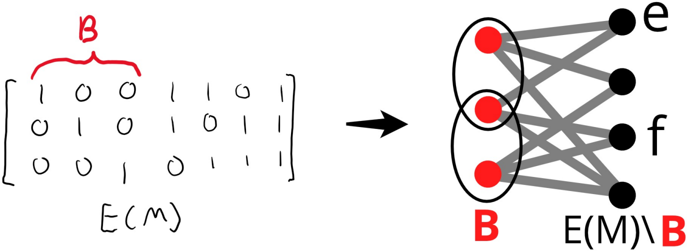

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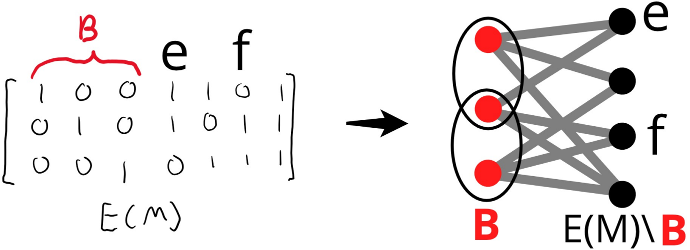

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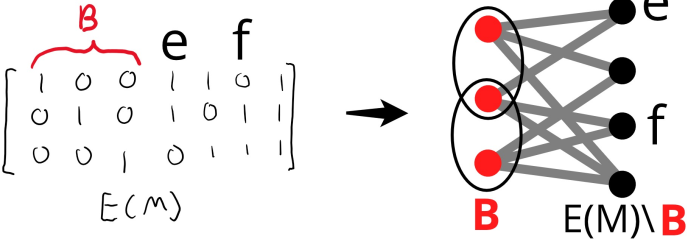

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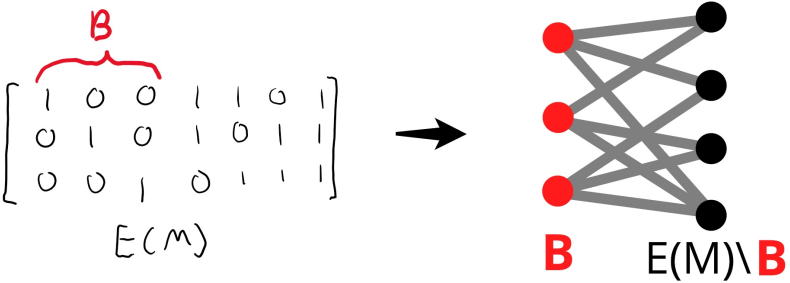
Graph classes with linear neighborhood complexity ($\eta \leq c \cdot m$)

- For each t and q, the fundamental graphs of GF(q)-representable matroids with no $M(K_t)$ or $M(K_t)^*$ minor.



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$U_{2,4}$

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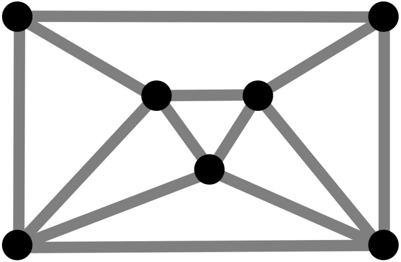

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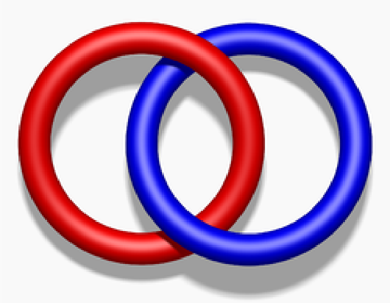

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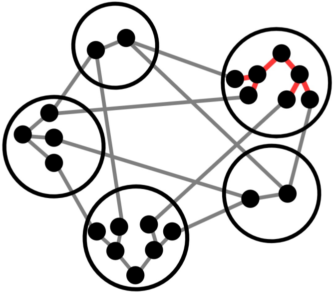

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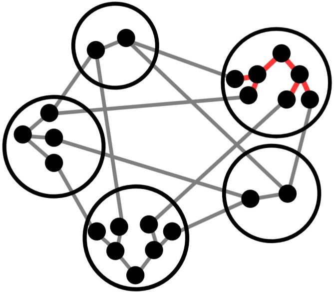

This idea can be used to characterize classes of **bounded expansion** [Reidl-Sánchez Villaamil-Stavropoulos 2019].

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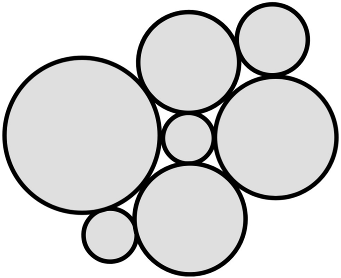

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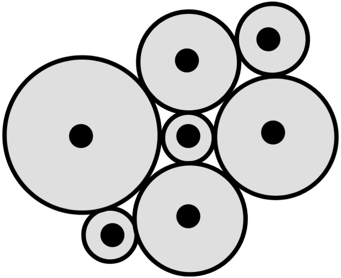

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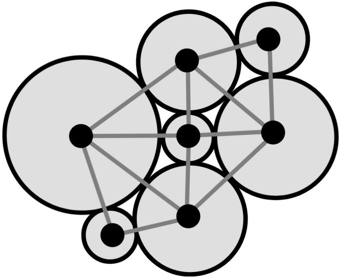

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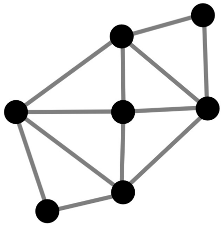

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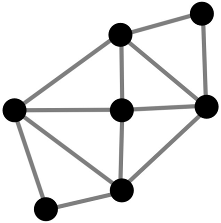

Every planar graph has a coin packing [Koebe–Andreev–Thurston].

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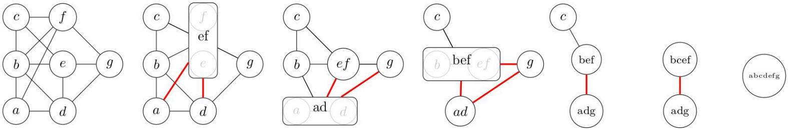

Introduced by [Bonnet-Kim-Thomassé-Watrigant 2021].

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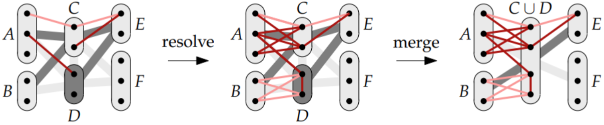

Introduced by [Dreier-Torunczyk 2025], proven by [Bonamy-Geniet 2025].

Theorem (DEMMPT 2024)

If $\mathcal{F}$ is a class such that **first-order logic** *cannot define all linear orders from $\mathcal{F}$, then for any G and any* **A** *of size m,*

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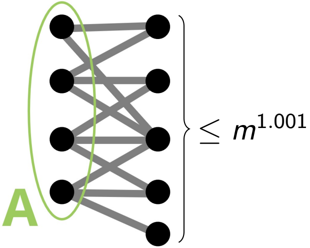

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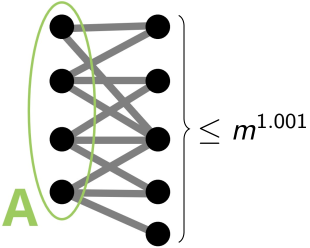

Conjecture (folklore)

The same bound holds if FO logic cannot define **all graphs**.

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In **first-order logic**, the language can:

- quantify over vertices ($\exists v \in V, \forall u \in V$),

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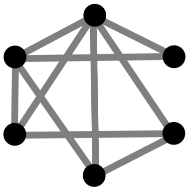

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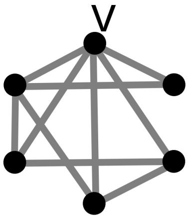

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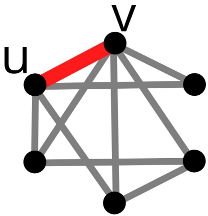

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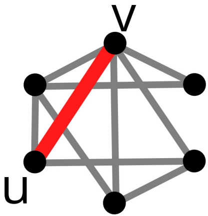

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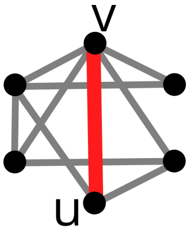

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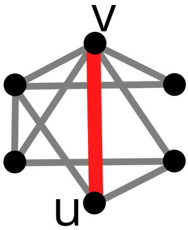

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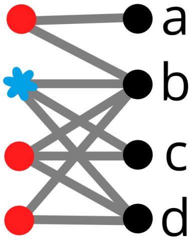
Theorem (Davies-Hatzel-Knauer-McCarty-Ueckerdt 2025+)

Any bipartite graph with neighborhood complexity $\leq c \cdot m$ for every m has **odd chromatic number** $\leq f(c)$.



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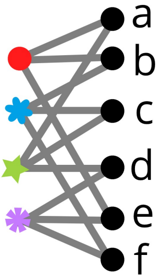

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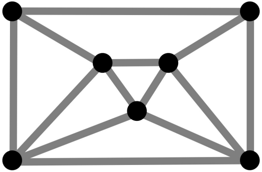

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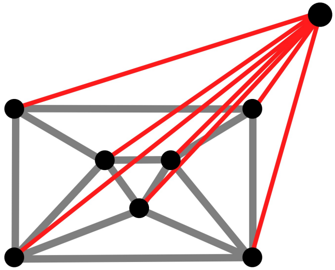

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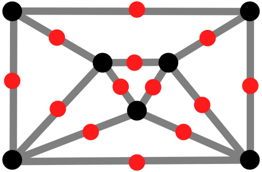

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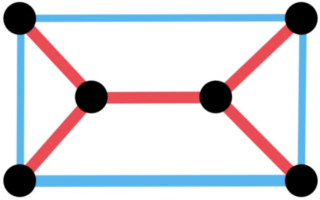

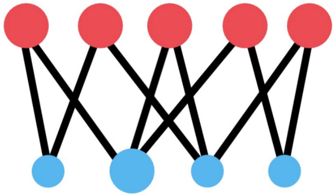

Question (DHKMU)

Does every fundamental graph of a planar graph have $\chi_{odd} \leq 5$*?*

Our bound: $\chi_{odd} \leq 98$.

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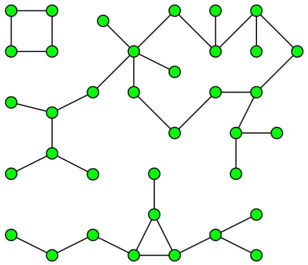

Conjecture (DHKMU)

Every cosimple matroid of girth $\geq f(t)$ contains at least one of $U_{t,2t}$, $B(K_t)$, $M(K_t)$, and $M(K_t)^*$ as a minor.

Thank you!