The first-order logic of graphs

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Tutte Colloquium

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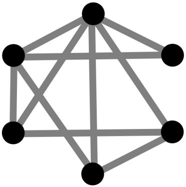
$$\phi = \exists v \in V : (\forall u \in V : (uEv \vee u = v)).$$



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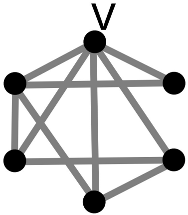

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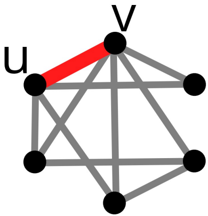

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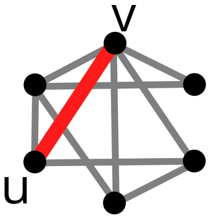

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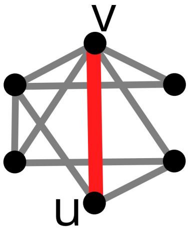

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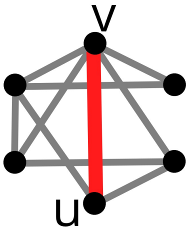

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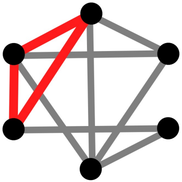
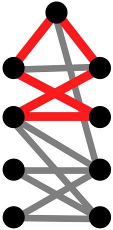
If ϕ is true for G, we write $G \models \phi$ and say G **models** ϕ.

Graph Properties

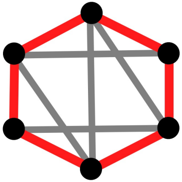
contains triangle



contains odd cycle



contains Hamiltonian cycle



$$G = (V, E)$$

Graph Properties

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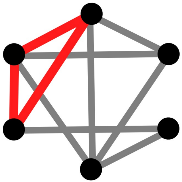
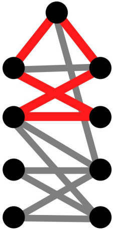


$\exists a, b, c \in V \ldots$

first-order

contains odd cycle



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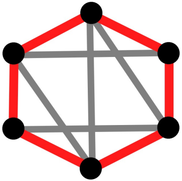

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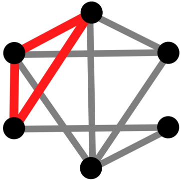

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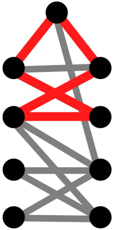
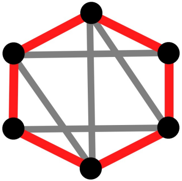


$\nexists X, Y \subseteq V \ldots$

$\mathbf{MSO_1}$

contains Hamiltonian cycle



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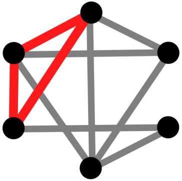

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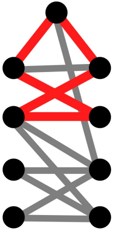

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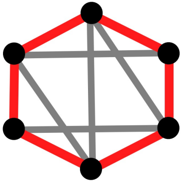


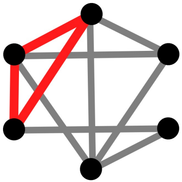
$\exists F \subseteq E \dots$

$\mathbf{MSO_2}$

$$G = (V, E)$$

Graph Properties

contains
triangle



$\exists a, b, c \in V \dots$

first-order

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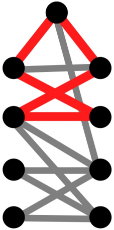
odd cycle



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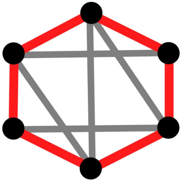


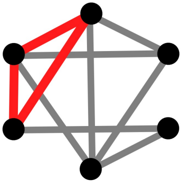
$\exists F \subseteq E \dots$

$\mathbf{MSO_2}$

Lemma: There is no $\mathbf{MSO_1}$ sentence ϕ_{Ham} which expresses "G contains a Hamiltonian cycle".

Graph Properties

contains triangle



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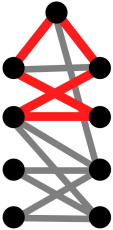
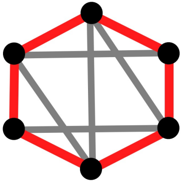


$\nexists X, Y \subseteq V \dots$

MSO$_1$

contains Hamiltonian cycle



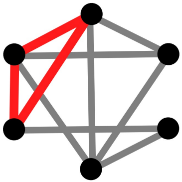
$\exists F \subseteq E \dots$

MSO$_2$

Lemma: There is no **first-order** sentence ϕ_{odd} which expresses "G contains an odd cycle".

Graph Properties

contains triangle



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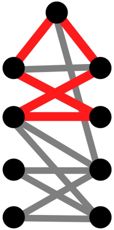

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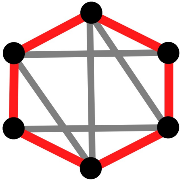

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$\mathbf{MSO_2}$

expressive power →

Graph Properties

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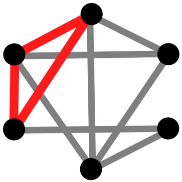

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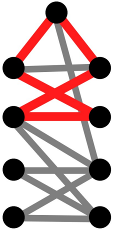

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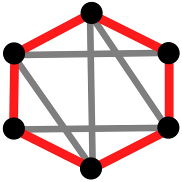

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MSO_2

expressive power →

← computational tractability

Graph Properties

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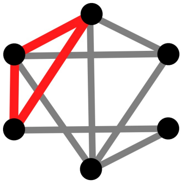

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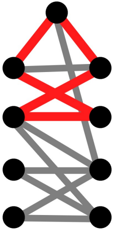

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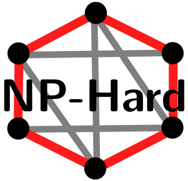

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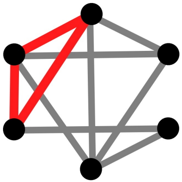

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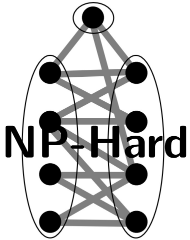
first-order

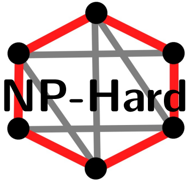
is 3-colorable



$\exists X, Y, Z \subseteq V \ldots$

MSO_1

contains Hamiltonian cycle



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MSO_2

expressive power →

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Graph Properties

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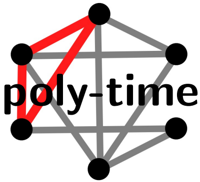

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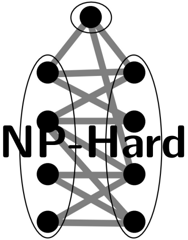

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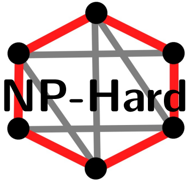

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MSO$_2$

expressive power →

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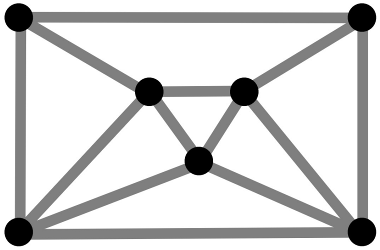

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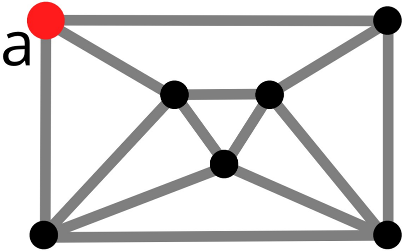

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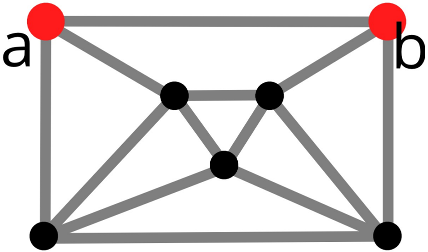

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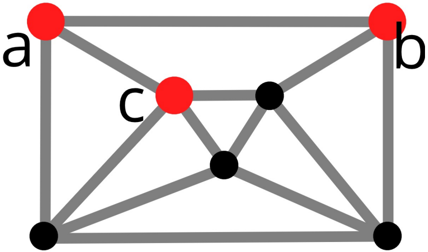

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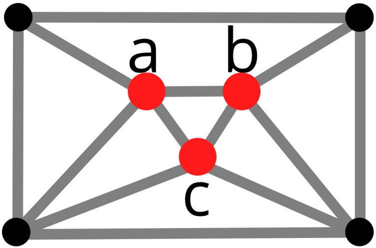

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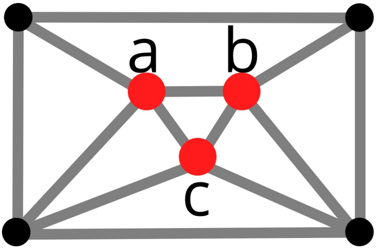
Naive takes time $\mathcal{O}(n^3)$.



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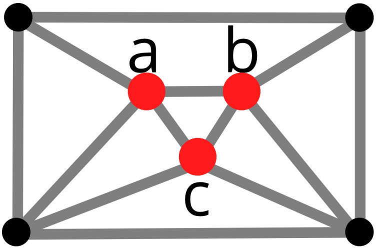

Matrix multiplication gives $\mathcal{O}(n^{2.372})$.

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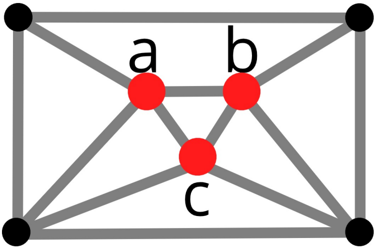

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Theorem (Dvořák, Král', Thomas 2011)

*Can check for triangles in time $\mathcal{O}(n)$ if G is **planar**.*

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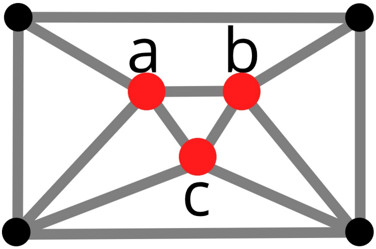

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Theorem (Dvořák, Král', Thomas 2011)

For each **first-order** *ϕ, can do in time $\mathcal{O}_\phi(n)$ if G is* **planar**.

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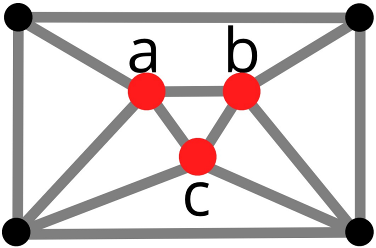

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Efficient: $\exists d$ s.t. we can check whether an n-vertex $G \in \mathcal{F}$ models an FO sentence ϕ in time $\mathcal{O}_\phi(n^d)$.

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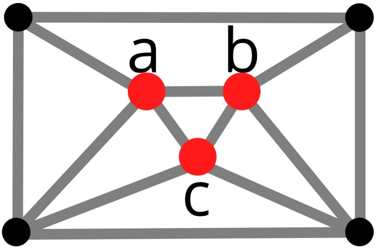

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i.e. $\mathcal{O}_\phi(n^6)$

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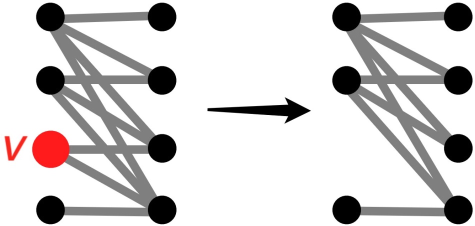

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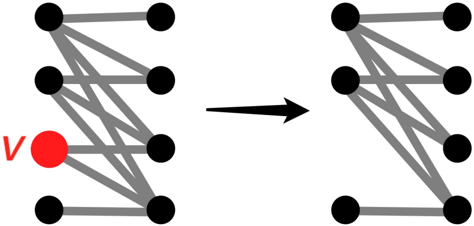

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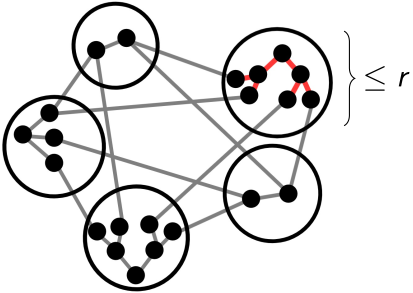

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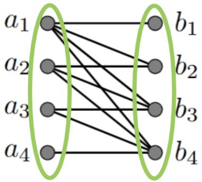

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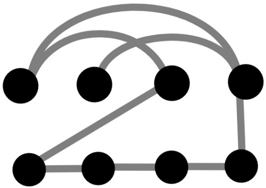

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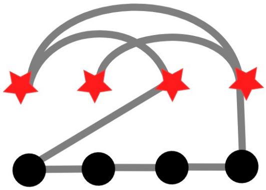

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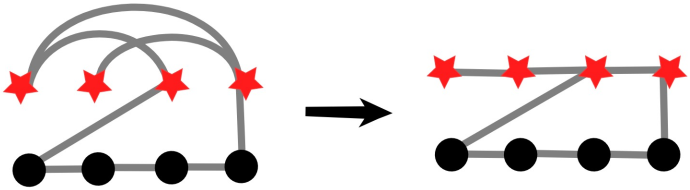

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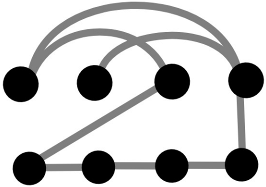

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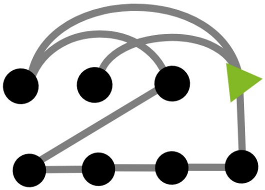

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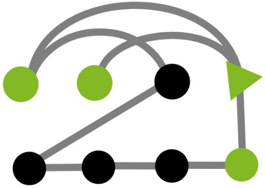

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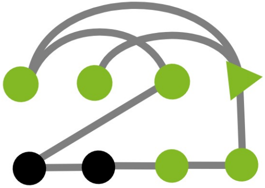

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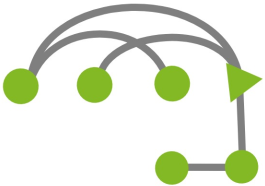

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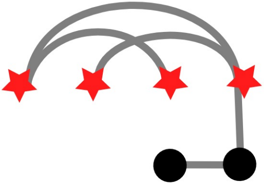

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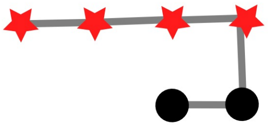

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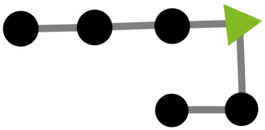

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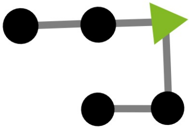

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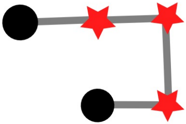

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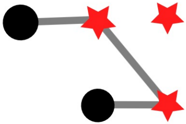

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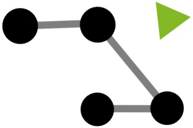

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Flipper **wins the game** on a class $\mathcal{F}$ if there exists $t = t(r)$ so that Flipper wins in $\leq t$ rounds on each $G \in \mathcal{F}$.

Next:

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2) The key ingredients to the algorithm.
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Theorem (DEMMPT 2024)

If $\mathcal{F}$ is **stable**, *then for any $G \in \mathcal{F}$ and any set* **A** *of n vertices,*

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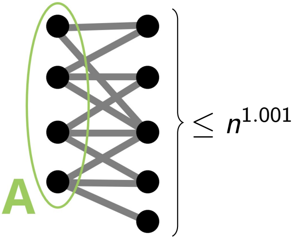

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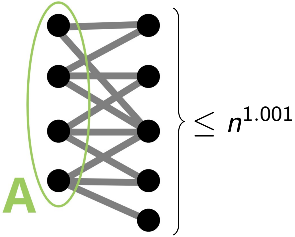

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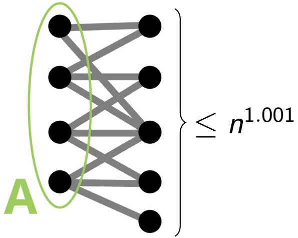

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Conjecture (Ossona de Mendez 2021)

Every **stable** *class is interdefinable with a* **sparse** *class.*

Thank you!