Graph theory tutorial

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Georgia Tech

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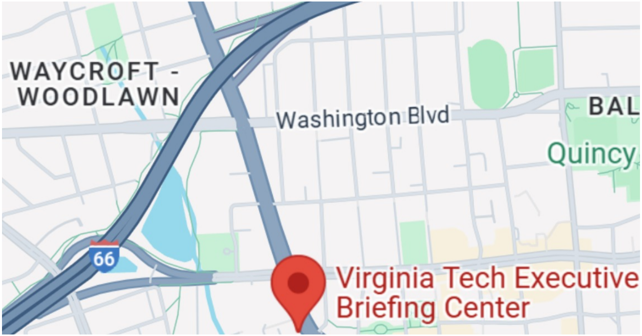
Classical graph theory



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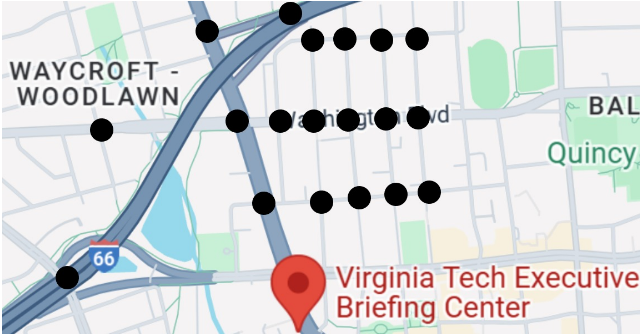
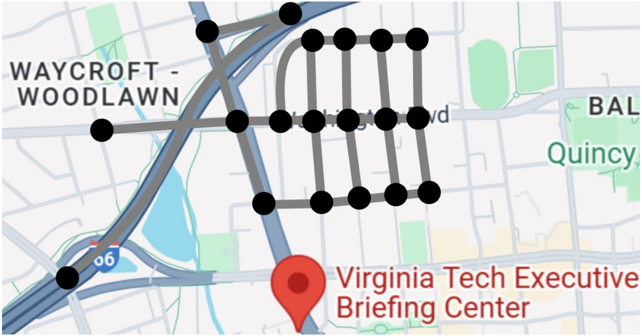


Vertices **V**

Classical graph theory



Vertices **V**
Edges **E**

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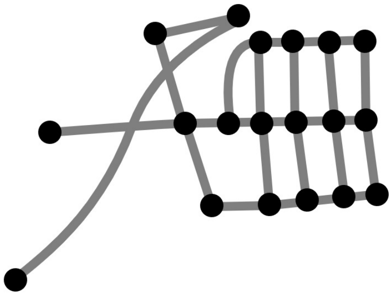

Vertices **V**
Edges **E**
Graph $G = (\mathbf{V}, \mathbf{E})$

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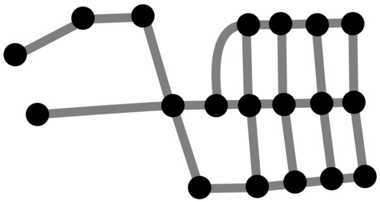

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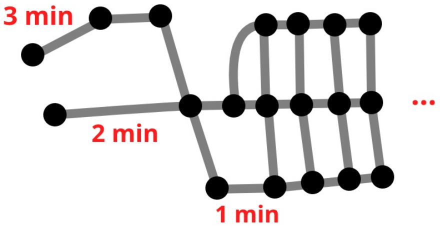
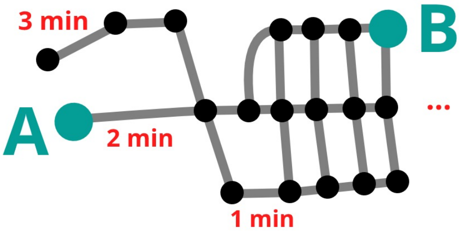


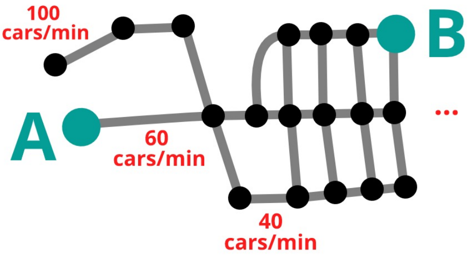
Vertices **V**
Edges **E**
Graph $G = (\mathbf{V}, \mathbf{E})$
Edge-weights $\mathbf{w} : \mathbf{E} \to \mathbb{R}_{\geq 0}$

Classical graph theory



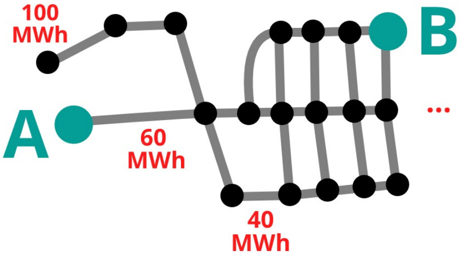
What is the fastest route from **A** to **B**?

Classical graph theory



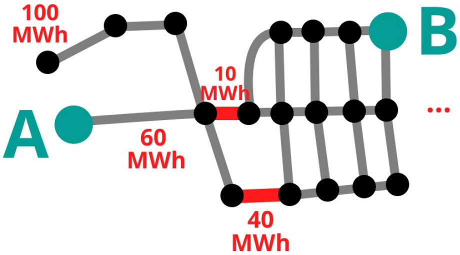
What is the fastest route from **A** to **B**?
How many cars can go from **A** to **B** without traffic?

Classical graph theory



What is the fastest route from **A** to **B**?
How much electricity can **A** send to **B**?

Classical graph theory



Theorem:
max-flow
=
min-cut

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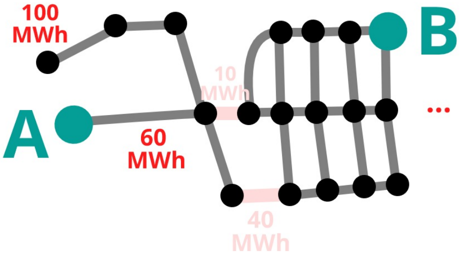

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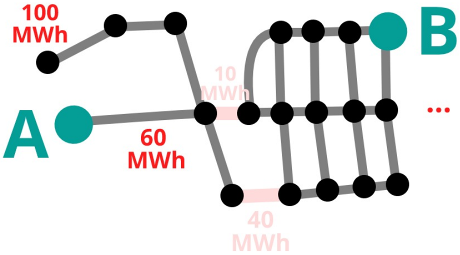

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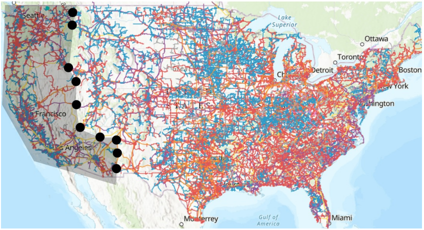
What is the fastest route from **A** to **B**?
How much electricity can **A** send to **B**?
Where is the power grid the most vulnerable?

Classical graph theory

A graph is a **bad expander** if deleting a few vertices/edges can disconnect a huge part.



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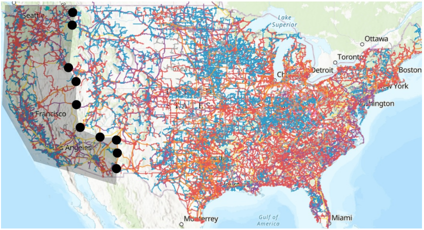

What is the fastest route from **A** to **B**?
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How can we design robust networks (ie **good expanders**)?

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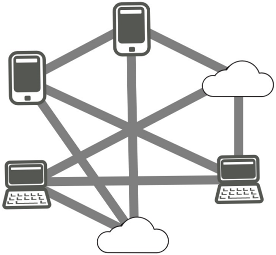

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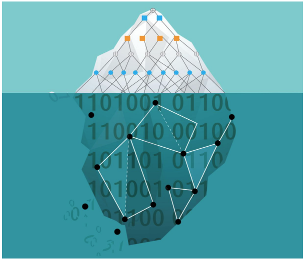

error correction, pseudorandomness, hash functions, Markov chains

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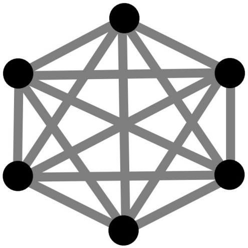

A **clique** is the best expander.

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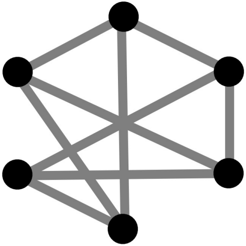

We understand optimal expanders where each vertex is incident to k edges.

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Theorem [LT]: Road networks without bridges are bad expanders.

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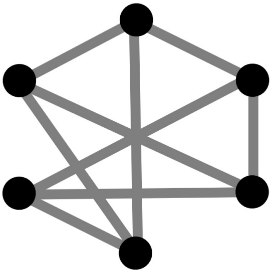
Keywords:
connectivity, separators, submodularity, linear programming

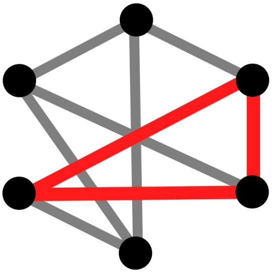
Modern trends in graph theory



1D-expander

How can we design **high-dimensional** expanders?

Modern trends in graph theory



2D-expander

How can we design **high-dimensional** expanders?

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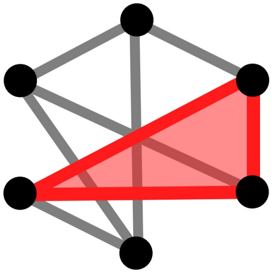

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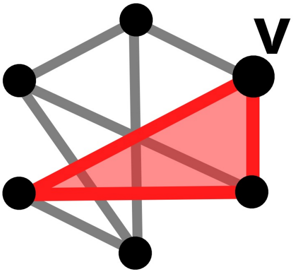

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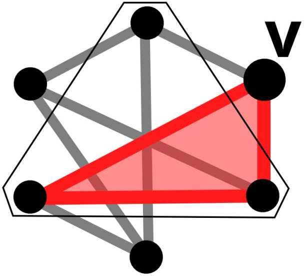

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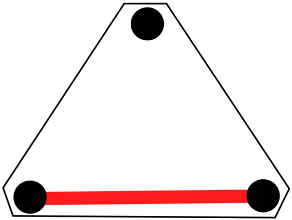

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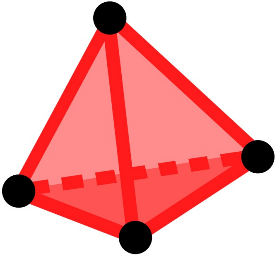

3D-expander

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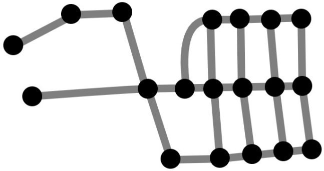
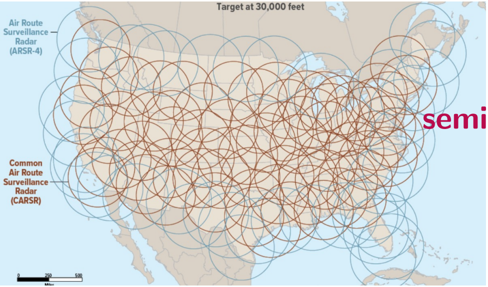


planar

How can we design **high-dimensional** expanders?
How can we determine and utilize **structural** properties?

Modern trends in graph theory



semi-algebraic

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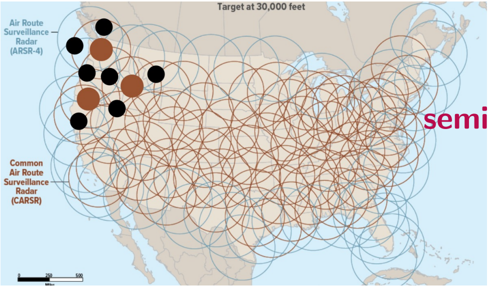

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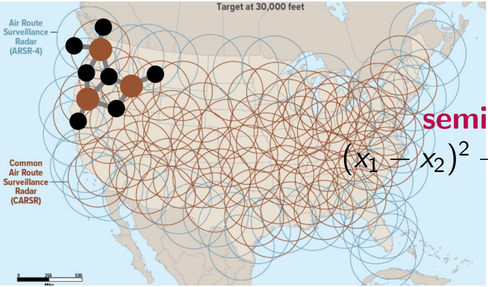

semi-algebraic

$(x_1 - x_2)^2 + (y_1 - y_2)^2 \leq r^2$

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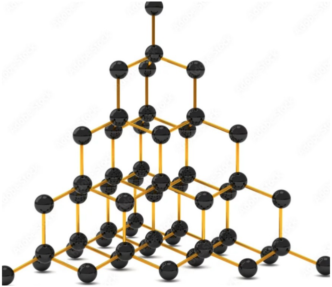

lattice
(diamond)

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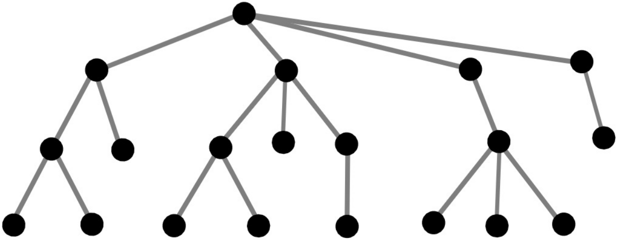

tree

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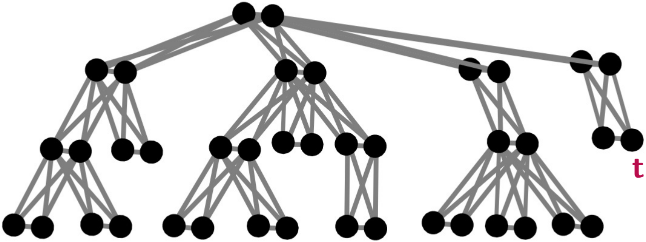

small **tree-width**

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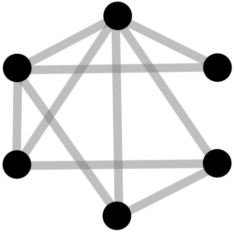
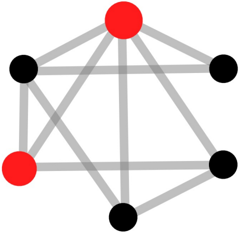


query complexity

How can we design **high-dimensional** expanders?
How can we determine and utilize **structural** properties?
Can **quantum computing** speed up classical graph problems?

Modern trends in graph theory



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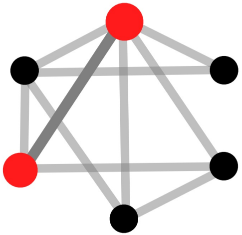

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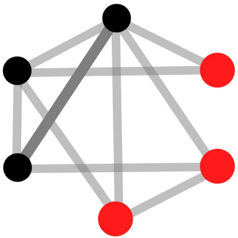

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$$\begin{array}{c} \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{array} \begin{array}{c} \begin{array}{ccc} U_1 & U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & ? \end{bmatrix} \end{array}$$

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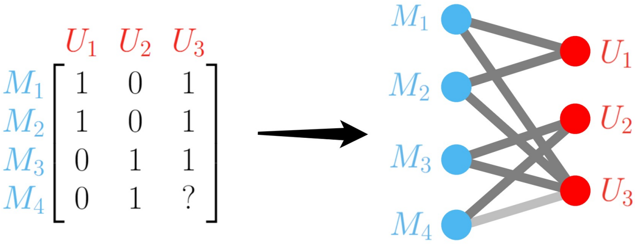

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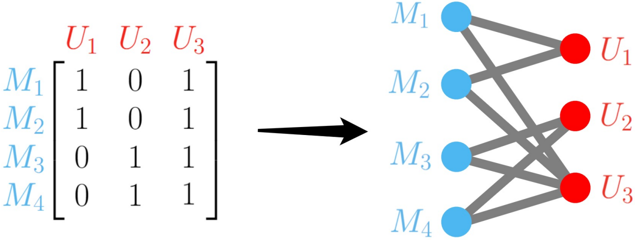
What if we care about the rank of **matrices**?

Modern trends in graph theory



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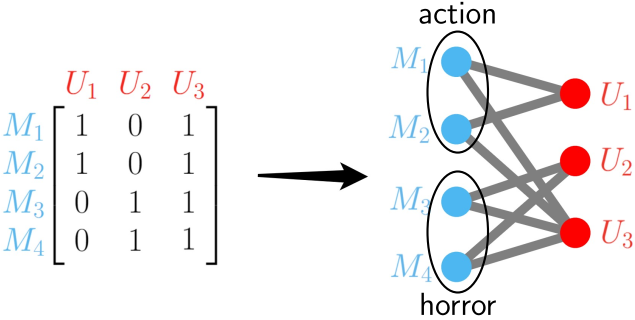

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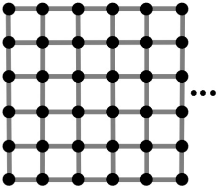
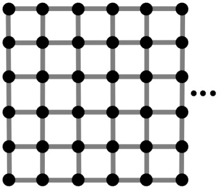
A graph-based model of quantum computing



(Raussendorf & Briegel 2001)

A graph-based model of quantum computing



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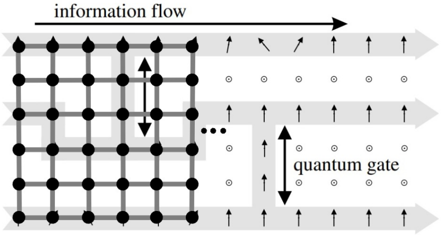

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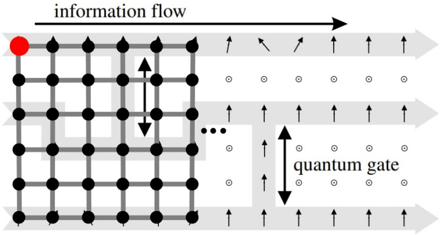

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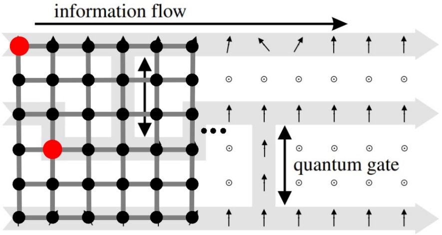

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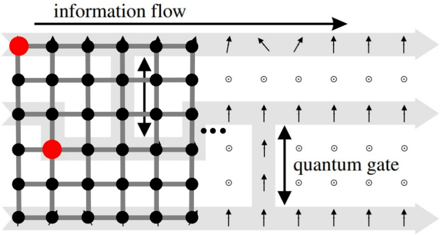

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Theorem: Roughly equivalent to quantum gate model.

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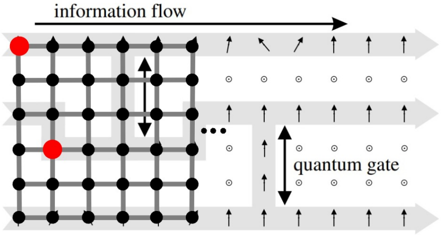

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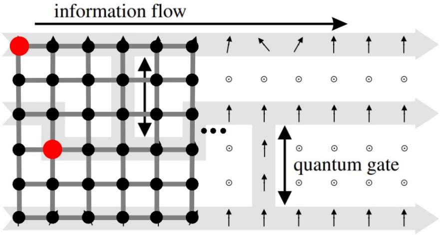

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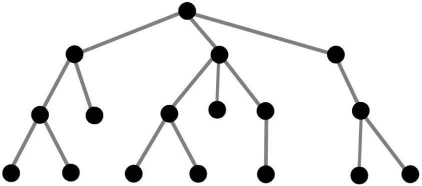
Conjecture (Geelen)**: All of them;** no proper fragment of quantum computing is stronger than classical computing.

A graph-based model of quantum computing

Theorem: True when G has small **rank-width**.



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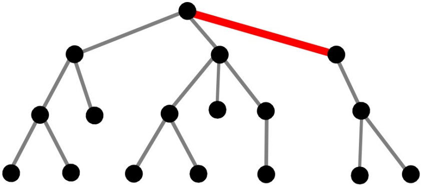

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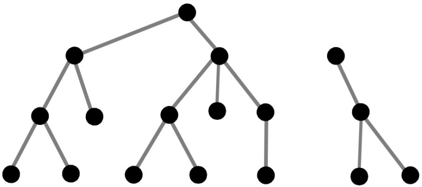
Theorem: True when G has small **rank-width**.



(Van den Nest, Dür, Vidal, & Briegel 2007)

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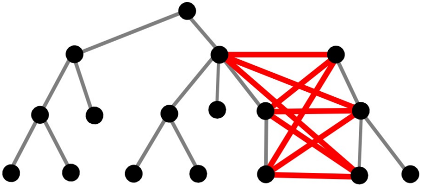

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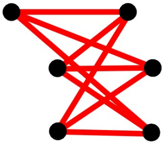

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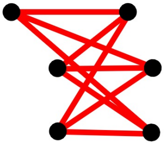

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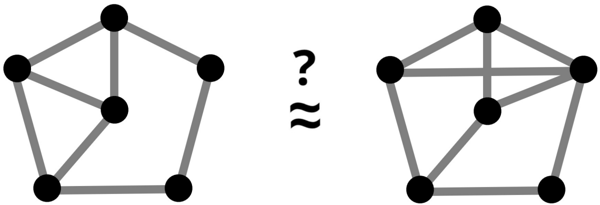

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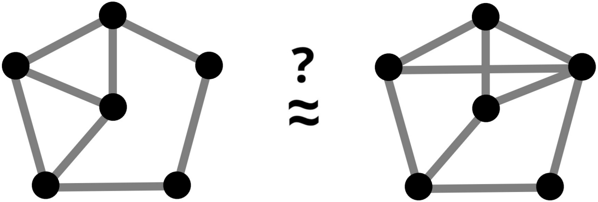

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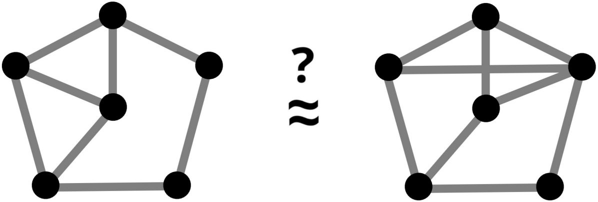

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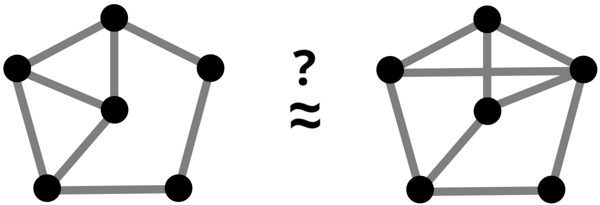

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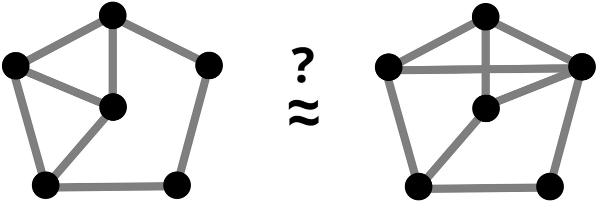

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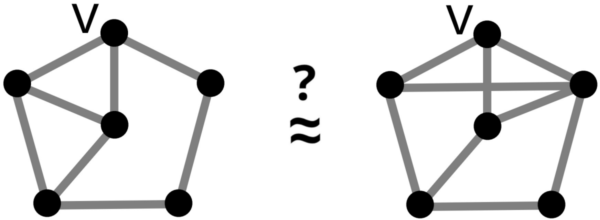

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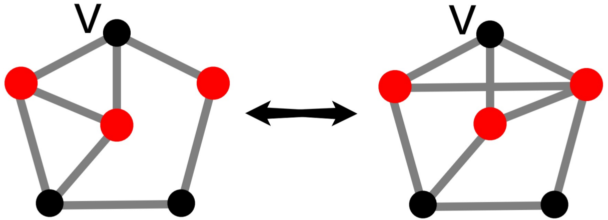

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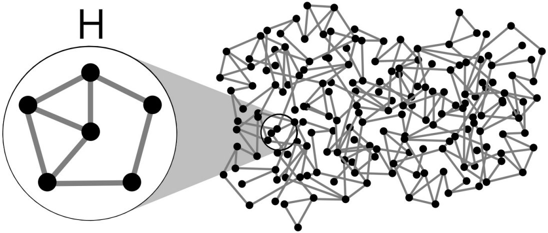
Forbidding a vertex-minor



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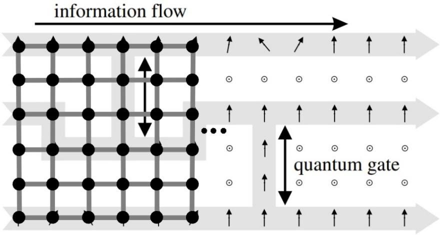

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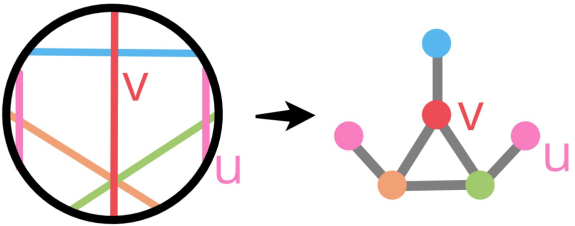

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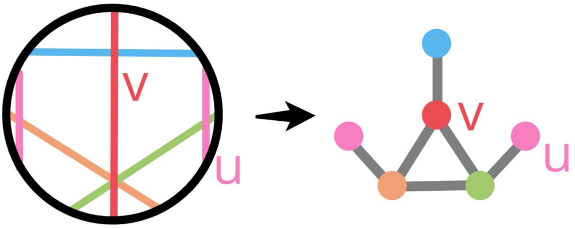

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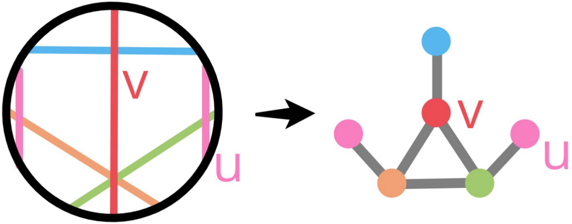

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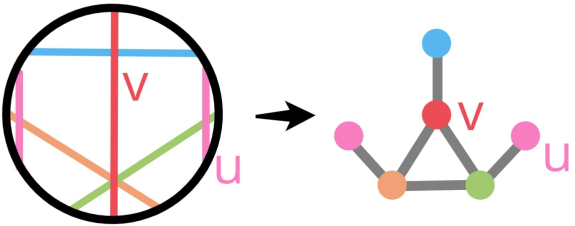

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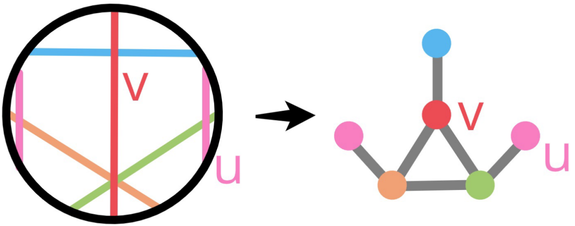

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Figures

- https://www.arcgis.com/apps/mapviewer/index.html?layers=d4090758322c4d32a4cd002ffaa0aa12
- https://simons.berkeley.edu/programs/summer-cluster-error-correcting-codes-high-dimensional-expansion
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- https://stock.adobe.com/images/diamond-crystal-structure-model-3d/44740015
- https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.86.5188

Thank you!