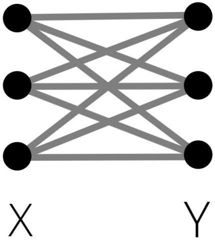
Connectivity for adjacency matrices and vertex-minors

Rose McCarty

Department of Combinatorics and Optimization
UNIVERSITY OF WATERLOO

Joint work with Jim Geelen and Paul Wollan



biclique

$$
\begin{array}{cc}
 & \begin{array}{cc} \text{X} & \text{Y} \end{array} \\
\begin{array}{c} \text{X} \\ \\ \text{Y} \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
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1 & 1 & 1 & 0 & 0 & 0 \\
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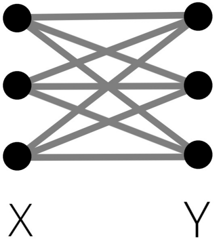
\end{bmatrix}
\end{array}
$$

adjacency matrix



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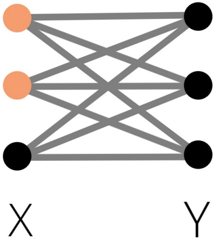

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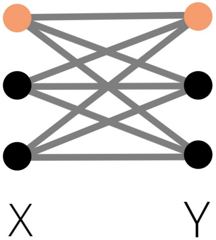

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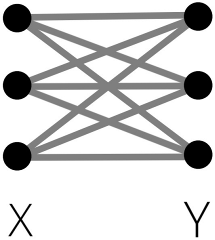
adjacency matrix

Matrices are over the binary field.



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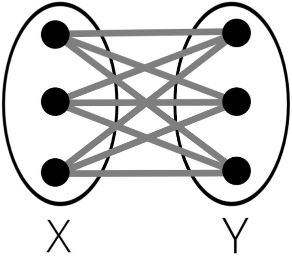
adjacency matrix

edge-connectivity $= \min_{X,Y} \#1\text{'s in } adj[X, Y]$



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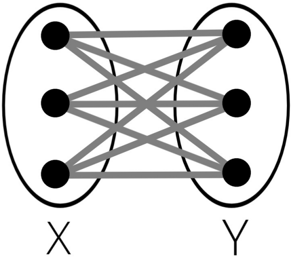
adjacency matrix

Rank(X, Y) is the rank of $adj[X, Y]$.



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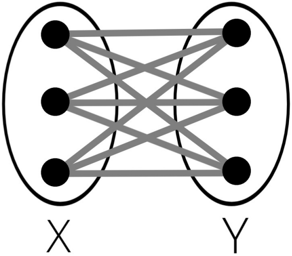

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adjacency matrix

$$\textbf{rank}(X, Y) = \textbf{rank}(Y, X)$$

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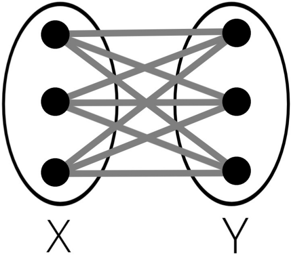

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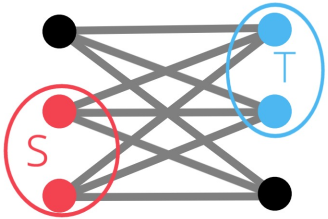
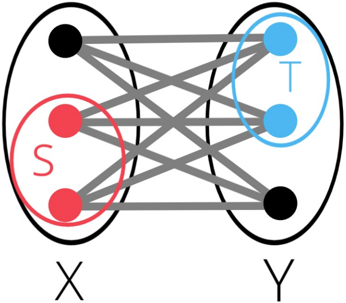
$$\textbf{rank}(X, Y) = \textbf{rank}(Y, X)$$

(Oum-Seymour)

rank-connectivity$(S, T) =$



$\textbf{rank-connectivity}(S, T) = \min_{X,Y} \textbf{rank}(X, Y)$



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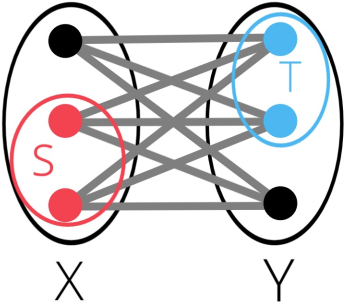
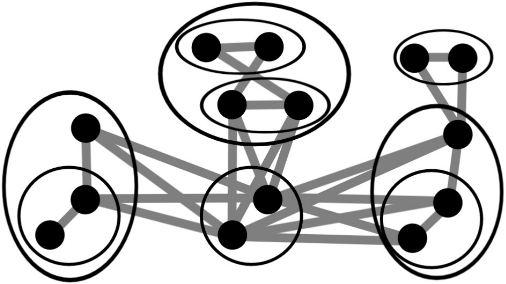


A graph is k-**rank-connected** if
rank$(X, Y) \geq \min(|X|, |Y|, k)$.

Why **rank-connectivity**?

- Good measure of complexity for dense graphs.



rank-width/clique-width, etc.

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(Menger)

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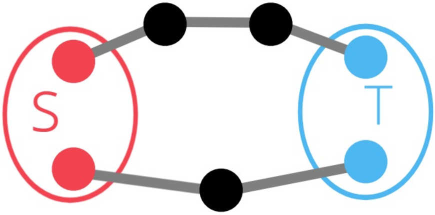

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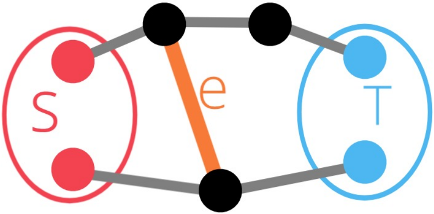

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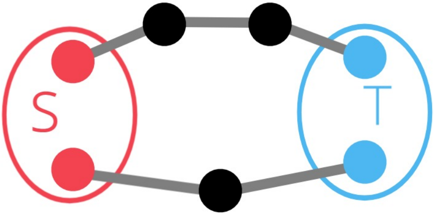

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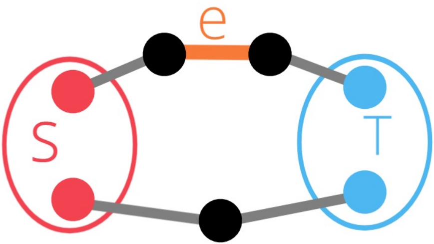

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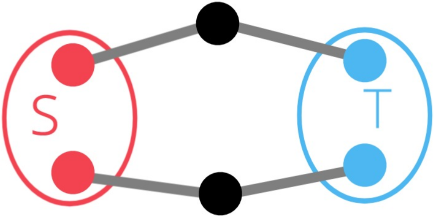

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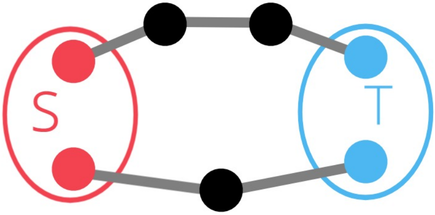

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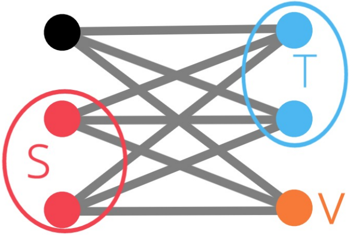

Most of the following maintain **rank-conn**(S, T):

$$G - v, \quad G \underline{*} v, \quad G \underline{\times} v. \qquad \text{(Oum)}$$

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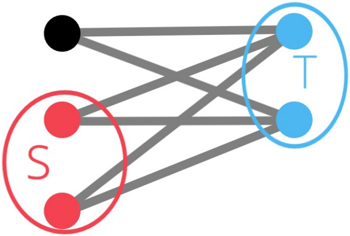

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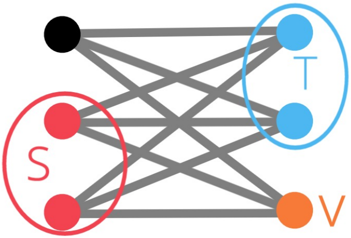

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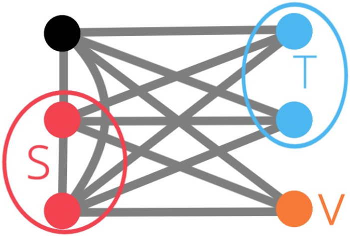

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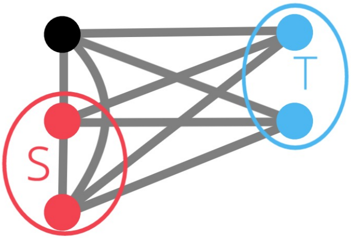

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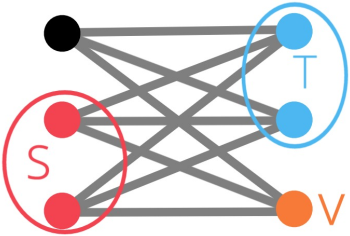

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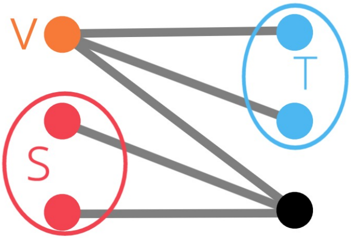

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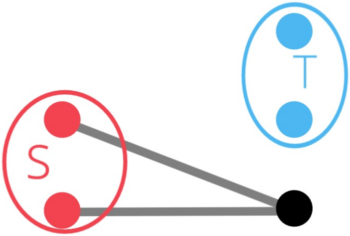

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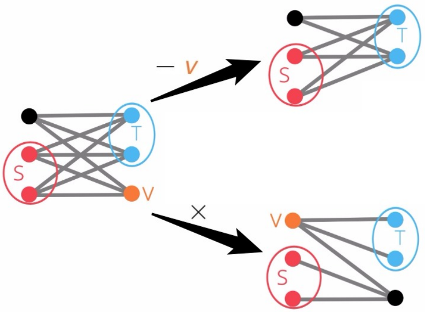
- **Pivot-minors** ($-v$ and $\times$) essentially generalize minors.



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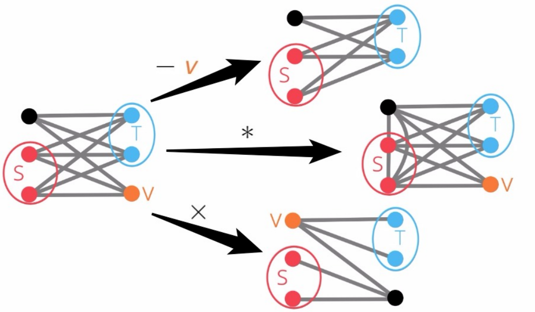

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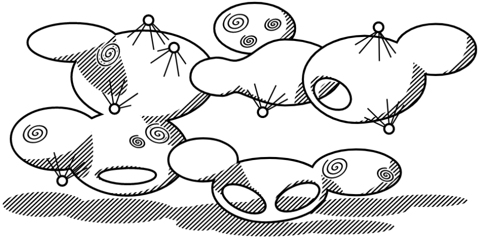
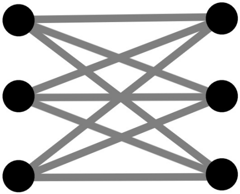


Figure by Felix Reidl

What is the structure of graphs with a forbidden **vertex-minor**?

When does every partition have **rank**$(X, Y) \leq k$?



G

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$adj(G)$

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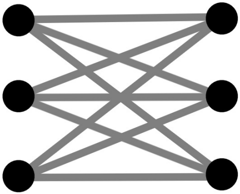

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One reason could be that the rank of $adj(G)$ is $\leq k$.

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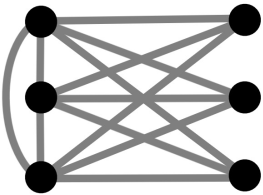

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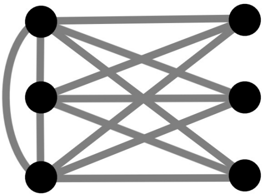

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$adj(G) + D$

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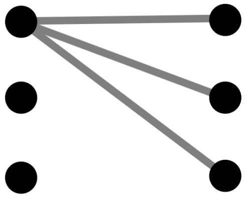

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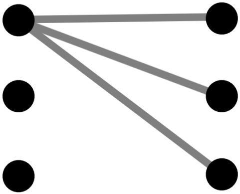

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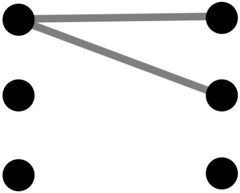
$adj(G)$

Theorem

If so, then there is a symmetric matrix M with $\leq f(k)$ non-zero entries s.t. the rank of $adj(G) + M + D$ is $\leq 2k$.

When does every partition have **rank**$(X, Y) \leq k$?



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$adj(G) + M$

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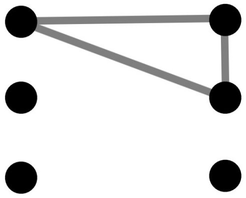


G

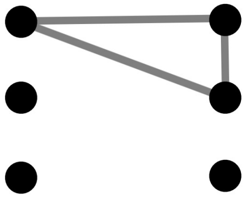
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$adj(G) + M$

Theorem

If so, then there is a symmetric matrix M with $\leq f(k)$ non-zero entries s.t. the rank of $adj(G) + M + D$ is $\leq 2k$.

When does every partition have **rank**$(X, Y) \leq k$?



G

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$adj(G) + M + D$

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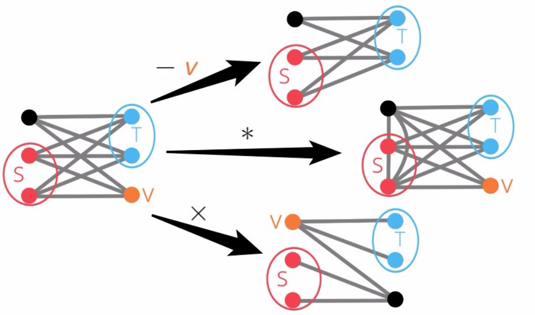


Theorem

*If so, then there is a symmetric matrix M with $\leq f(k)$ non-zero entries s.t. $adj(G) + M$ is a k-**perturbation** of 0.*

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

rank 3

$$\underset{\text{rank 3}}{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}} = \underset{\text{rank 1}}{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} + \underset{\text{rank 2}}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}$$

$$\underset{\text{rank 3}}{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}} = \underset{\text{rank 1}}{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} + \underset{\text{rank 2}}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}$$

$$\vec{v}\vec{v}^{\mathsf{T}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

rank 3 rank 1 rank 2

$$\vec{v}\vec{v}^{\mathsf{T}} \qquad \vec{u}\vec{a}^{\mathsf{T}} + \vec{a}\vec{u}^{\mathsf{T}}$$

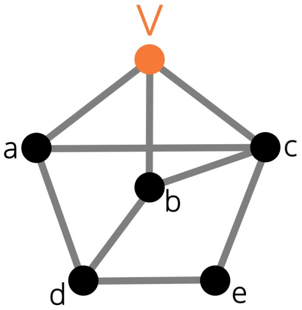
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

rank 3 | rank 1 | rank 2

$\vec{v}\vec{v}^{\mathsf{T}}$ | $\vec{u}\vec{a}^{\mathsf{T}} + \vec{a}\vec{u}^{\mathsf{T}}$

$*$ | $\times$

Locally complementing $(*)$ at v replaces the induced subgraph on the neighborhood of v by its complement.

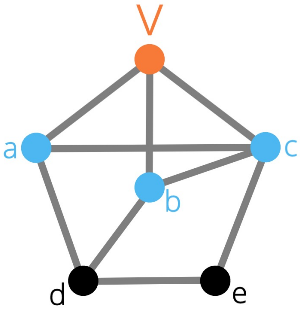


G

	v	a	b	c	d	e
v	0	1	1	1	0	0
a	1	0	0	1	1	0
b	1	0	0	1	1	0
c	1	1	1	0	0	1
d	0	1	1	0	0	1
e	0	0	0	1	1	0

$\mathrm{adj}(G)$

Locally complementing $(*)$ at v replaces the induced subgraph on the neighborhood of v by its complement.

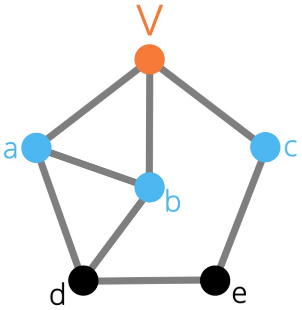


G

	v	a	b	c	d	e
v	0	1	1	1	0	0
a	1	0	0	1	1	0
b	1	0	0	1	1	0
c	1	1	1	0	0	1
d	0	1	1	0	0	1
e	0	0	0	1	1	0

adj(G)

Locally complementing ($*$) at v replaces the induced subgraph on the neighborhood of v by its complement.

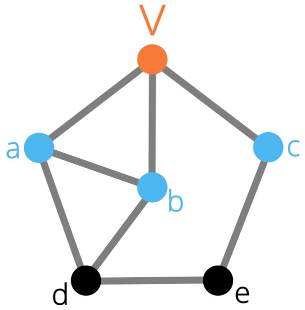


$G * v$

$$\begin{array}{c|cccccc} & v & a & b & c & d & e \\ \hline v & 0 & 1 & 1 & 1 & 0 & 0 \\ a & 1 & 1 & 1 & 0 & 1 & 0 \\ b & 1 & 1 & 1 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 1 & 0 & 1 \\ d & 0 & 1 & 1 & 0 & 0 & 1 \\ e & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

$\text{adj}(G) + \vec{v}\vec{v}^\intercal$

Locally complementing $(*)$ at v replaces the induced subgraph on the neighborhood of v by its complement.



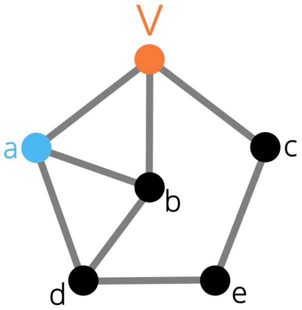
$G * v$

$$\begin{array}{c} \\ v \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccccc} v & a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\mathrm{adj}(G) + \vec{v}\vec{v}^{\mathsf{T}}$

Rank(X, Y) is the same in G and $G * v$.

Pivoting ($\times$) on an edge va complements between three sets and exchanges labels

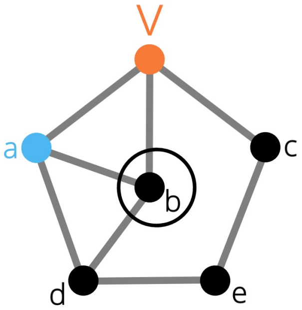


G

$$\begin{array}{c} \\ v \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccccc} v & a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\text{adj}(G)$

Pivoting ($\times$) on an edge va complements between three sets and exchanges labels.

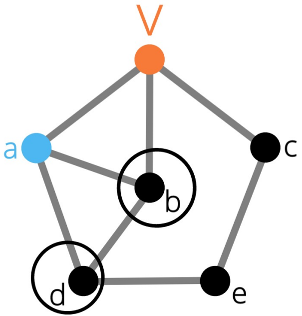


G

	v	a	b	c	d	e
v	0	1	1	1	0	0
a	1	0	1	0	1	0
b	1	1	0	0	1	0
c	1	0	0	0	0	1
d	0	1	1	0	0	1
e	0	0	0	1	1	0

adj(G)

Pivoting ($\times$) on an edge va complements between three sets and exchanges labels.

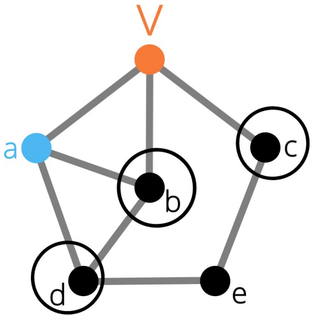


G

	v	a	b	c	d	e
v	0	1	1	1	0	0
a	1	0	1	0	1	0
b	1	1	0	0	1	0
c	1	0	0	0	0	1
d	0	1	1	0	0	1
e	0	0	0	1	1	0

adj(G)

Pivoting $(\times)$ on an edge va complements between three sets and exchanges labels.

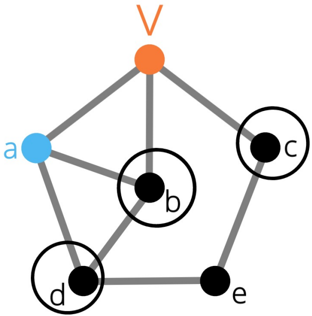


G

$$\begin{array}{c} \\ v \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccccc} v & a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\text{adj}(G)$

Pivoting ($\times$) on an edge va complements between three sets and exchanges labels.

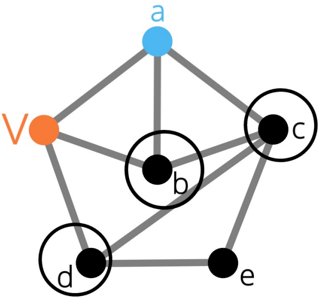


G

	v	a	b	c	d	e
v	1	1	1	1	0	0
a	1	1	1	0	1	0
b	1	1	0	0	1	0
c	1	0	0	0	0	1
d	0	1	1	0	0	1
e	0	0	0	1	1	0

$\widetilde{\text{adj}}(G)$

Pivoting ($\times$) on an edge va complements between three sets and exchanges labels.

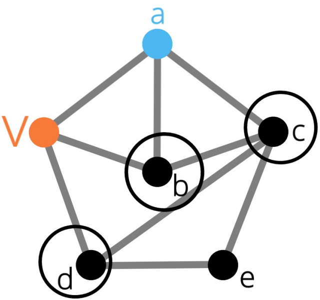


$$G \times va$$

	v	a	b	c	d	e
v	1	1	1	0	1	0
a	1	1	1	1	0	0
b	1	1	0	1	0	0
c	0	1	1	0	1	1
d	1	0	0	1	0	1
e	0	0	0	1	1	0

$$\widetilde{\mathrm{adj}}(G) + \vec{v}\vec{a}^{\mathsf{T}} + \vec{a}\vec{v}^{\mathsf{T}}$$

Pivoting ($\times$) on an edge va complements between three sets and exchanges labels; $G \times va = G * v * a * v = G * a * v * a$.



$G \times va$

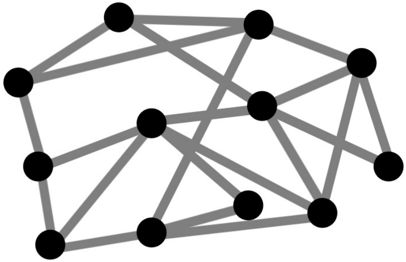
	v	a	b	c	d	e
v	1	1	1	0	1	0
a	1	1	1	1	0	0
b	1	1	0	1	0	0
c	0	1	1	0	1	1
d	1	0	0	1	0	1
e	0	0	0	1	1	0

$\widetilde{\mathrm{adj}}(G) + \vec{v}\vec{a}^{\mathsf{T}} + \vec{a}\vec{v}^{\mathsf{T}}$

Rank(X, Y) is the same in G and $G \times va$.

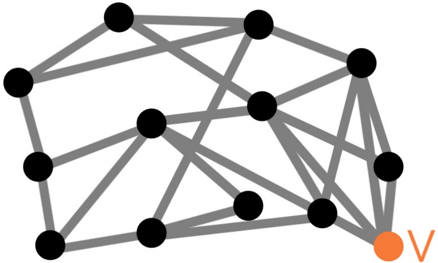
A k-**perturbation** G' of G is obtained by adding k vertices and then removing them:

1) $G - v$
2) $G \underline{*} v$
3) $G \underline{\times} v$



A ***k*-perturbation** G' of G is obtained by adding k vertices and then removing them:

1) $G - v$
2) $G \underline{*} v = G * v - v$
3) $G \underline{\times} v$



A **k-perturbation** G' of G is obtained by adding k vertices and then removing them:

1) $G - v$
2) $G \underline{*} v = G * v - v$
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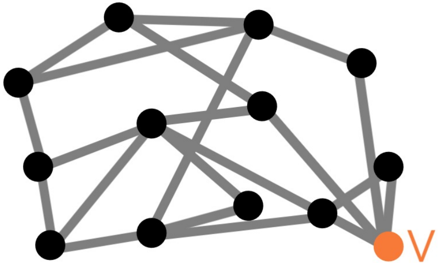

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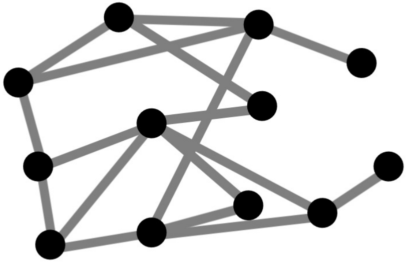

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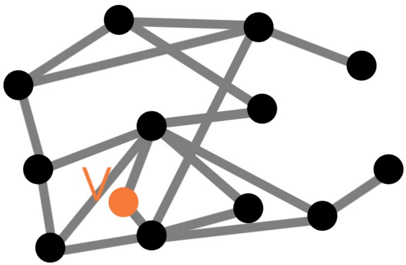

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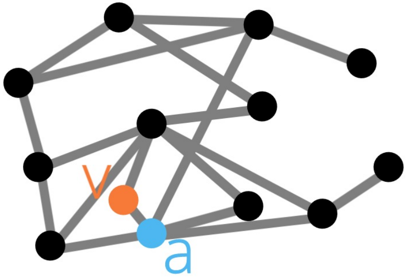

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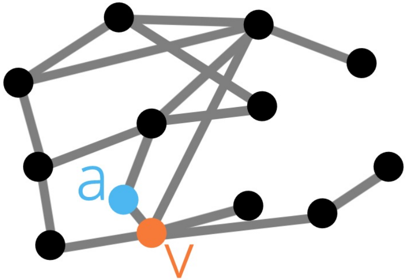

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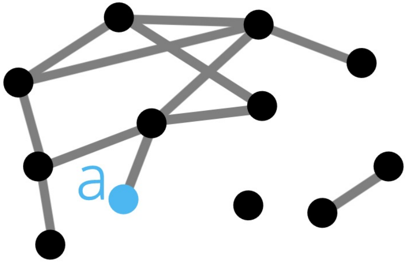

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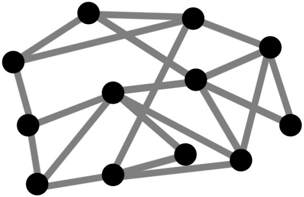
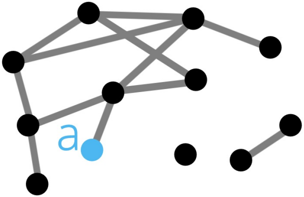
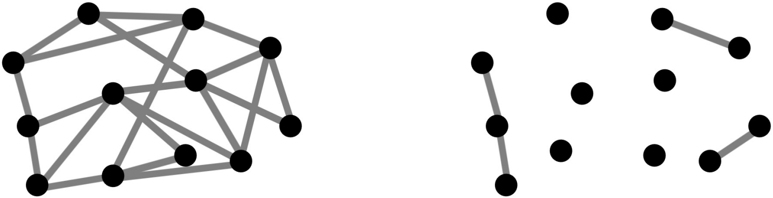




$$\mathbf{rank}_G(X, Y) \leq \mathbf{rank}_{G'}(X, Y) + k$$

A ***k*-perturbation** G' of G is obtained by adding k vertices and then removing them:

1) $G - v$
2) $G \mathbin{\underline{*}} v = G * v - v$
3) $G \mathbin{\underline{\times}} v = G \times va - v$



Theorem

If every partition has **rank**$(X, Y) \leq k$*, then there exists a* ***k*-perturbation** *of* G *with* $\leq f(k)$ *edges.*

A ***k*-perturbation** G' of G is obtained by adding k vertices and then removing them:

1) $G - v$
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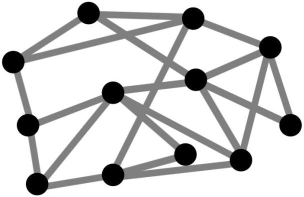
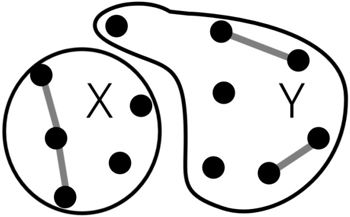


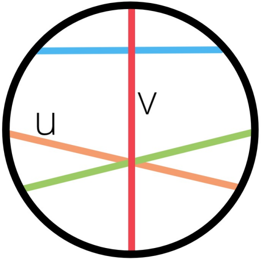


Theorem

If every partition has **rank**$(X, Y) \leq k$*, then there exists a* ***k*-perturbation** *of* G *with* $\leq f(k)$ *edges.*

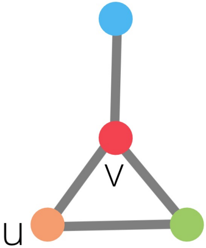
Conjecture

Every r_H-**rank-connected** *graph with no* H-**vertex-minor** *is a* k_H-**perturbation** *of an* **intersection graph** *of chords on a circle.*



chords on a circle

$\longrightarrow$



intersection graph

Thank you!