Decomposing a graph into odd trails

Jim Geelen, Rose McCarty, and Paul Wollan

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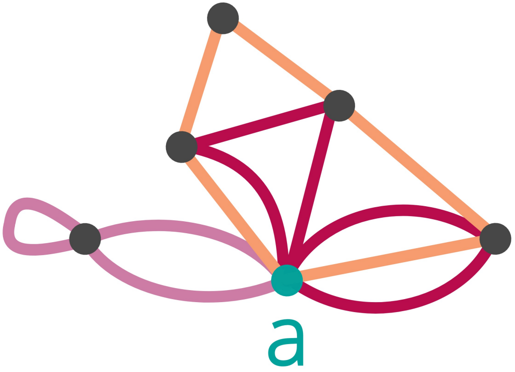
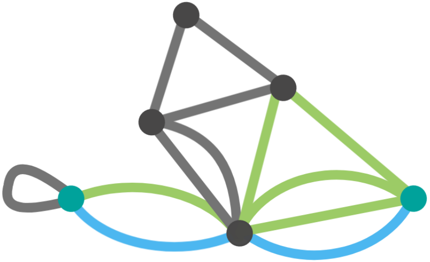
When can the edge set of a graph be decomposed into k odd a-trails?



Immersion of

H

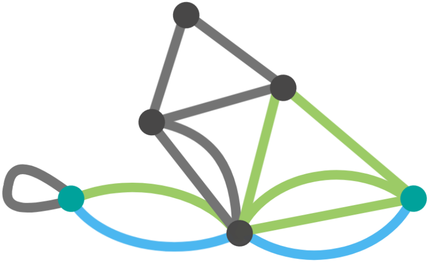
into



G

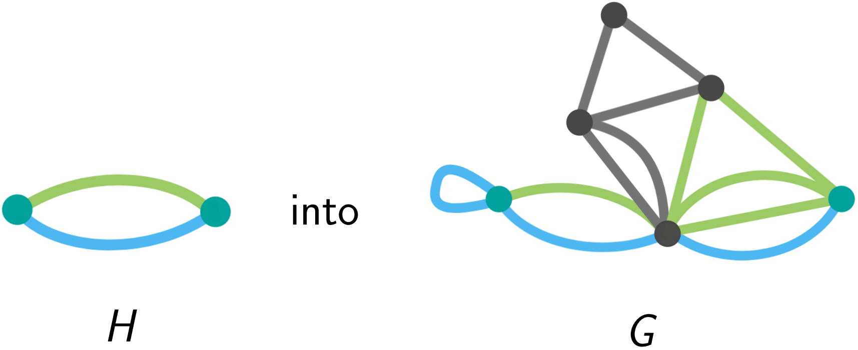
Flooding...

into

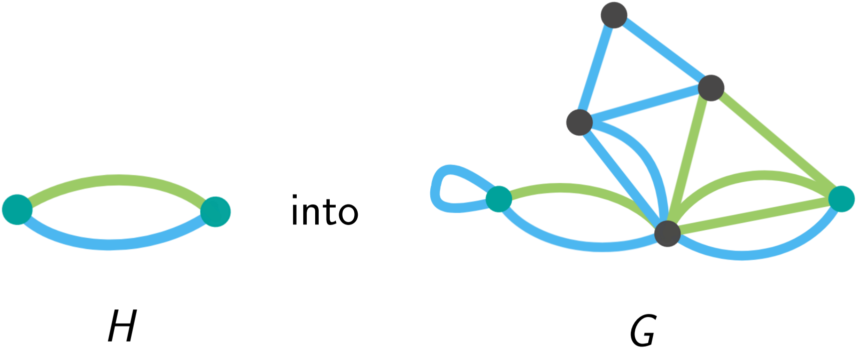


H *G*

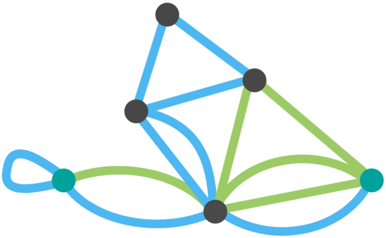
Flooding...



Flooding...

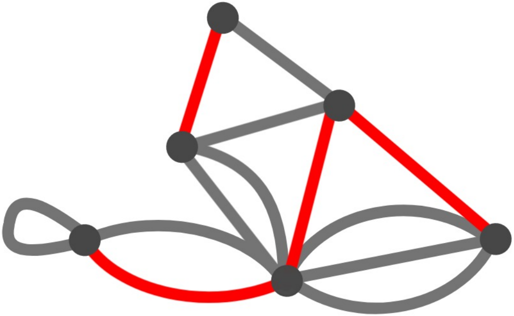


floods



H G

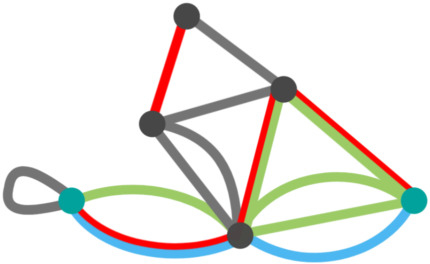
Signed Graphs



$$(G, \Sigma)$$

Immersion of

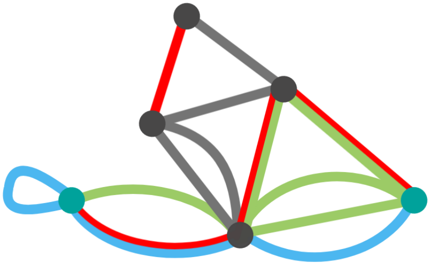
into



(H, Σ_H) (G, Σ)

Flooding...

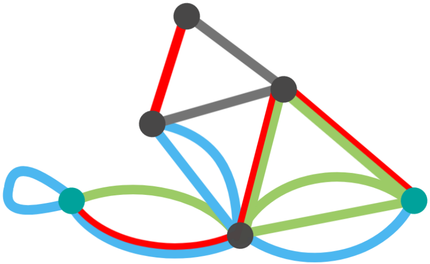
into



(H, Σ_H) (G, Σ)

Flooding...

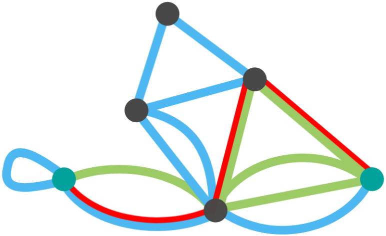
into



(H, Σ_H) (G, Σ)

(H, Σ_H)

floods



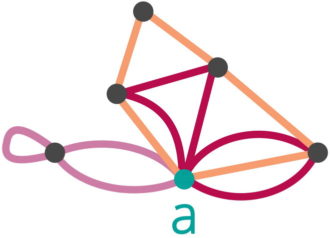
(G, Σ')

$E(G)$ decomposes into k odd a-trails

$\longleftrightarrow$

$(H, E(H))$

floods



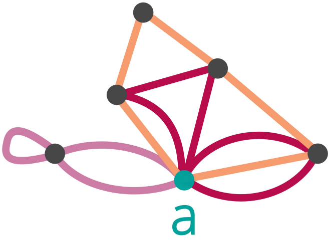
$(G, E(G))$

with v sent to a.

Let $\nu(G, \Sigma, a)$ denote maximum k so that

$(H, E(H))$

floods

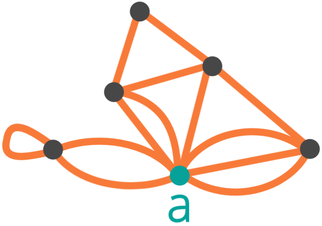


(G, Σ)

with v sent to a.

If $\nu(G, \Sigma, a) = k$ then

floods



$(H, E(H))$

(G, Σ)

with v sent to a.

Structure of...

- graphs with no K_t-immersion
  (DeVos, McDonald, Mohar, Scheide 12; Wollan 13)
- signed graphs with no $(K_t, E(K_t))$-immersion
  (Churchley and Mohar 18)

Problem

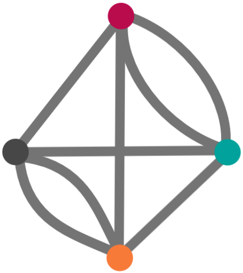
Describe the structure of signed graphs that are not **flooded** *by* $(K_t, E(K_t))$.

Motivation: structure of graphs with a forbidden vertex minor

Theorem (Geelen, Kwon, McCarty, Wollan)

For each circle graph H, every graph of sufficiently large rank-width has a graph isomorphic to H as a vertex minor.

$\longrightarrow$



Structure of...

- graphs with no K_t-immersion (DeVos, McDonald, Mohar, Scheide 12; Wollan 13)
- signed graphs with no $(K_t, E(K_t))$-immersion (Churchley and Mohar 18)

Corollary

If G is internally 4-edge-connected, then there exists a set of at most $3\nu(G, \Sigma, a)$ edges whose deletion removes all odd a-trails.

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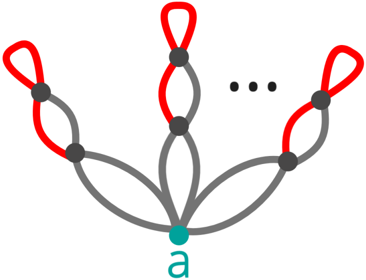

(G, Σ)

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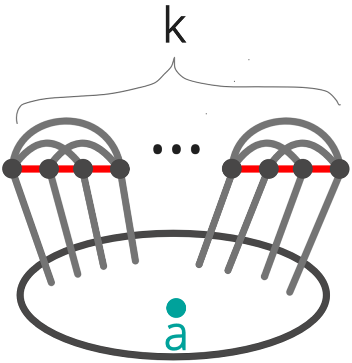
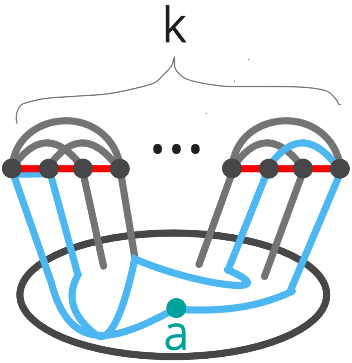


$$\nu(G, \Sigma, a) \leq k$$

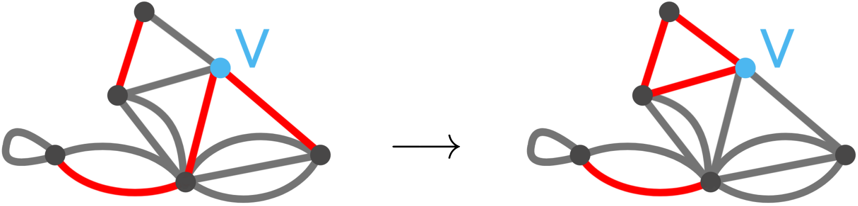
Corollary

If G is internally 4-edge-connected, then there exists a set of at most $3\nu(G,\Sigma,a)$ edges whose deletion removes all odd a-trails.



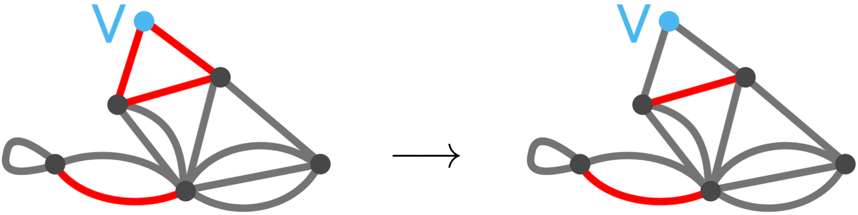
$$\nu(G,\Sigma,a) \leq k$$

Re-signing at v



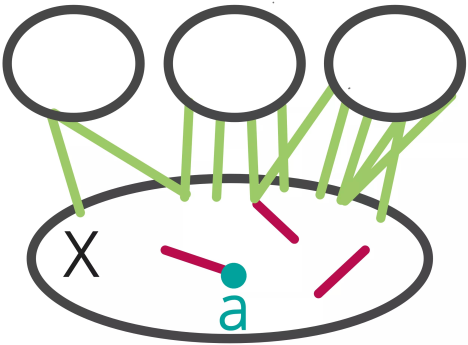
(G, Σ) $\qquad\qquad$ (G, Σ')

Re-signing at v



(G, Σ') $\longrightarrow$ (G, Σ'')

$$\nu(G, \Sigma, a) \leq \min_{\Sigma', X} |\Sigma'(X)| + \frac{|\delta(X)|}{2}$$



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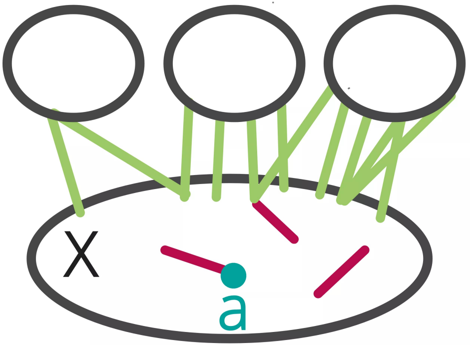
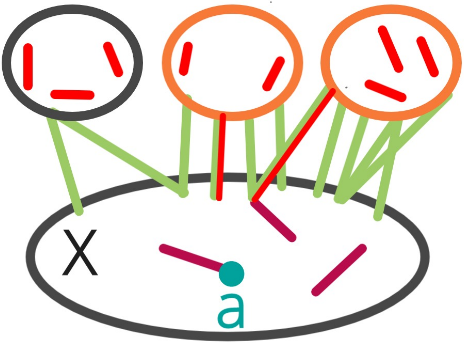


Equality holds if trails are not required to flood. (Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour 04)

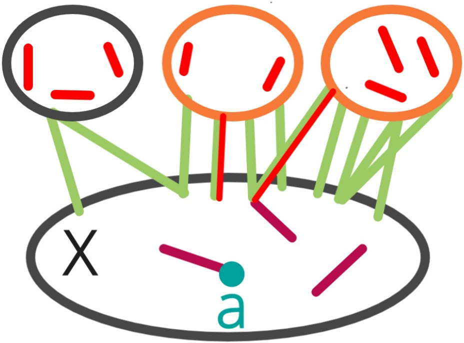
$$\nu(G,\Sigma,a) \leq \min_{\Sigma',X} |\Sigma'(X)| + \frac{|\delta(X)|}{2}$$



A component C of $G - X$ is **odd** if

$$|\Sigma'(C)| + |\Sigma'(\delta(C))| \not\equiv \frac{|\delta(C)|}{2} \mod 2.$$

$$\nu(G,\Sigma,a) \leq \min_{\Sigma',X} |\Sigma'(X)| + \frac{|\delta(X)|}{2} - \mathrm{odd}(G,\Sigma',X)$$



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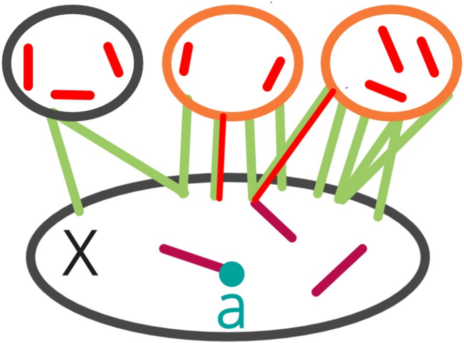
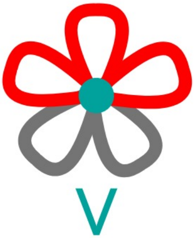
Main Theorem

For all signed graphs (G, Σ) *and* $a \in V(G)$,

$$\nu(G, \Sigma, a) = \min_{\Sigma', X} \Sigma'(X) + \frac{|\delta(X)|}{2} - odd(G, \Sigma', X).$$



Notice that (G, Σ) floods (H, Σ_H):



with v sent to a

where $\deg_H(v) = \deg_G(a)$

and $|\Sigma_H| = \nu(G, \Sigma, a)$.

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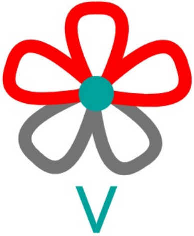
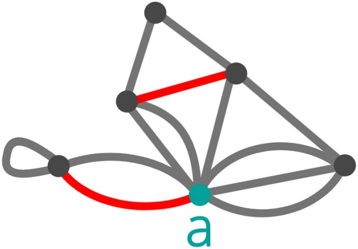


with v sent to a

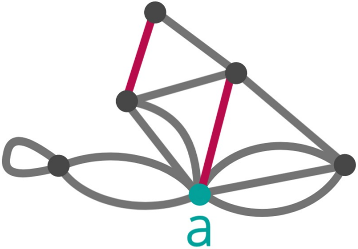
where $\deg_H(v) = \deg_G(a)$

and $|\Sigma_H| = \nu(G, \Sigma, a)$.

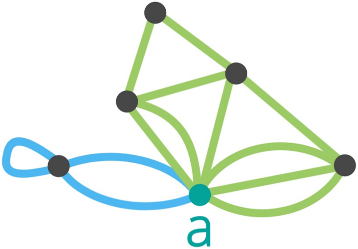
Define a matroid of rank $|E(H) \setminus \Sigma_H|$.



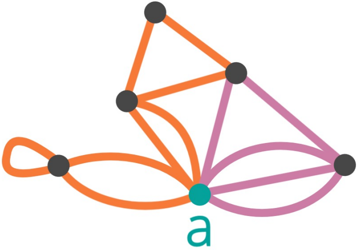
(G, Σ_1)



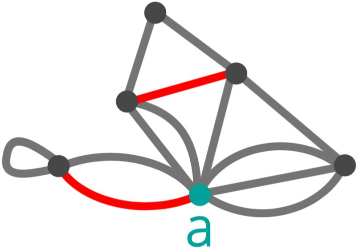
(G, Σ_2)



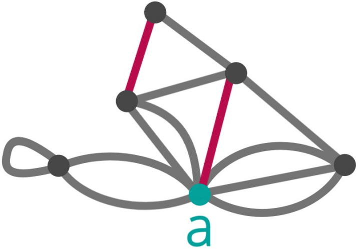
$\nu(G, \Sigma_1, a) = 2$



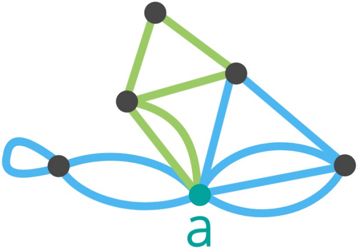
$\nu(G, \Sigma_2, a) = 2$



(G, Σ_1)



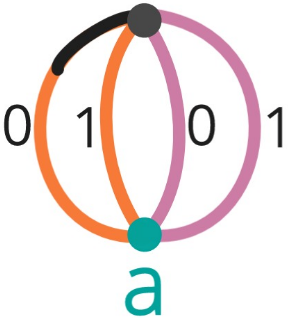
(G, Σ_2)



All trails are odd for both Σ_1 and Σ_2.

Theorem

If $\nu(G, \Sigma_1, a) = \nu(G, \Sigma_2, a) = k$*, then* $E(G)$ *decomposes into* k *trails, all of which begin and end at* a *and are odd for* **both** Σ_1 *and* Σ_2.



Theorem

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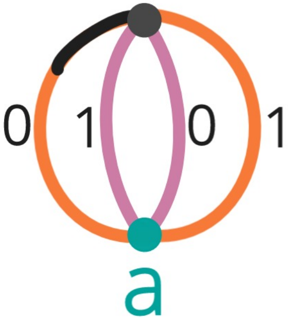

- Flooding by $(K_t, E(K_t))$
- Group-labelled graphs
- Erdös-Posá property for flooding by *a*-trails in a collection satisfying exchange property