# Average degree and bicliques

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UNIVERSITY OF WATERLOO

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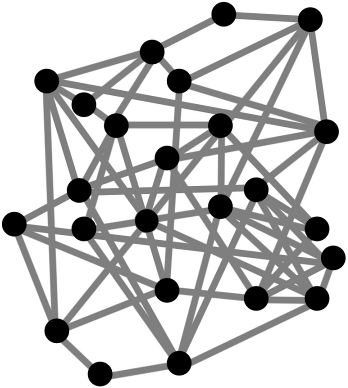
Combinatorics Seminar at Birmingham

When is the maximum average degree of a graph tied to the size of its largest balanced biclique?



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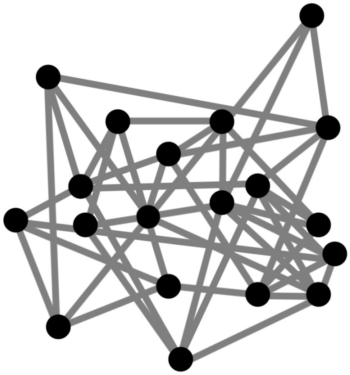

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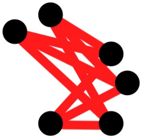

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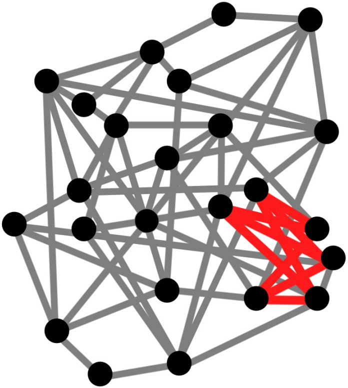
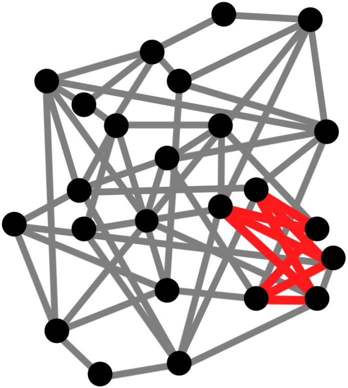


**Biclique number** $\tau(G) :=$
maximum $t$ so that $G$ has $K_{t,t}$-subgraph

For which classes of graphs does there exist a function $f$ so that $\mathrm{mad}(G) \leq f(\tau(G))$?



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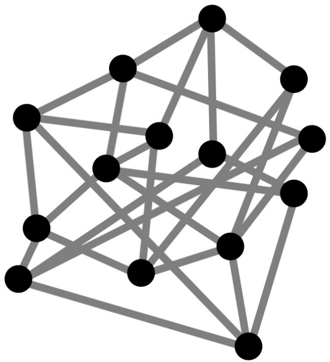

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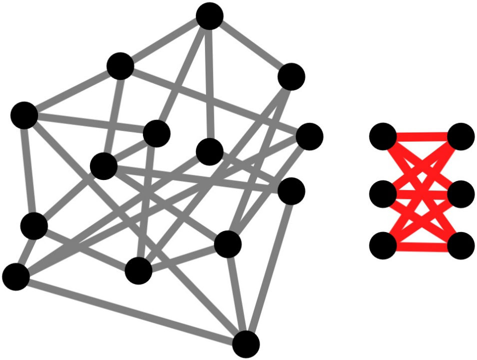

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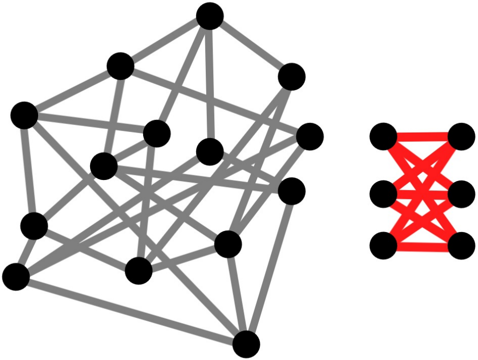

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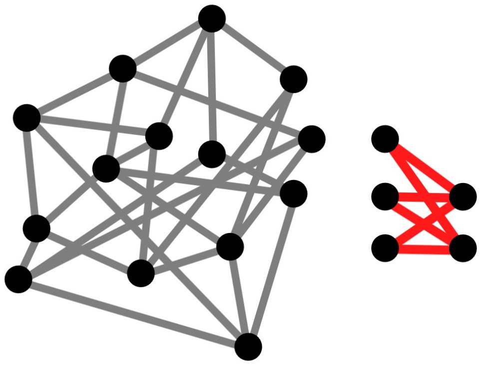

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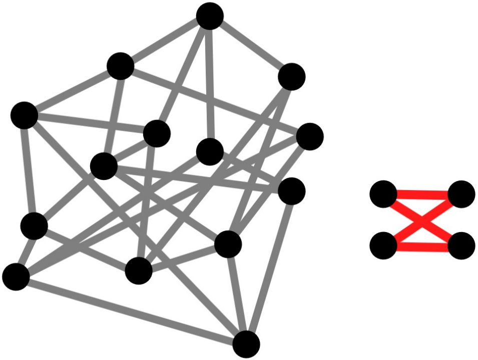

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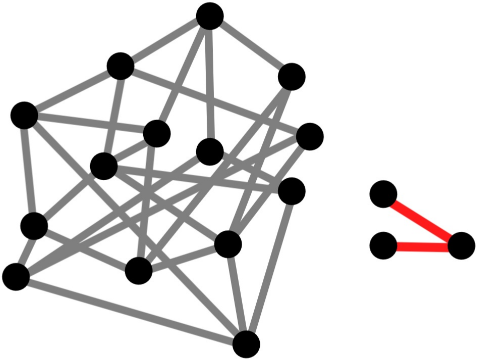

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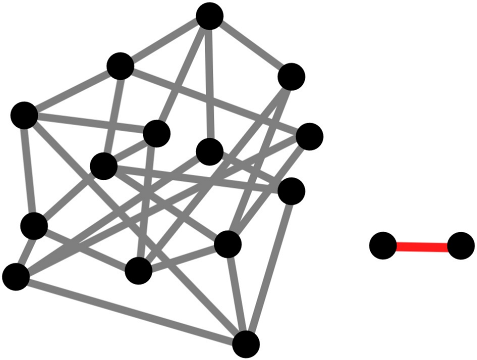

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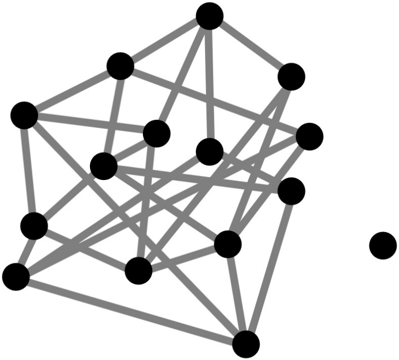

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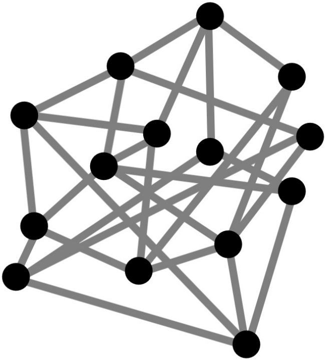

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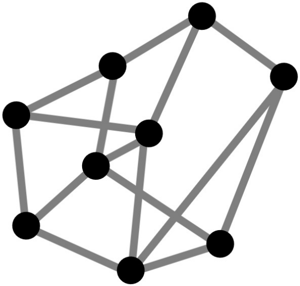
For which classes of graphs does there exist a function $f$ so that $\mathrm{avgdeg}(G) \leq f(\tau(G))$?



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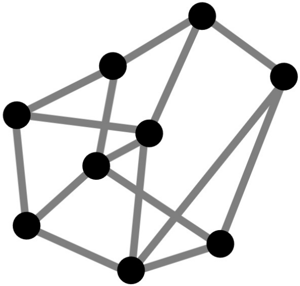

Call such a class **degree-bounded** and $f$ a **bounding function**.

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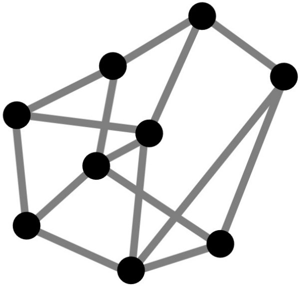

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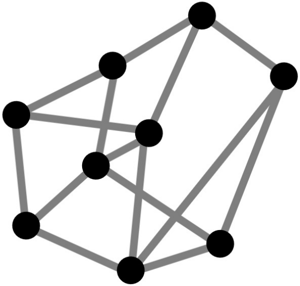

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Thomassen conjectured this for **subgraph-closed** classes 83.

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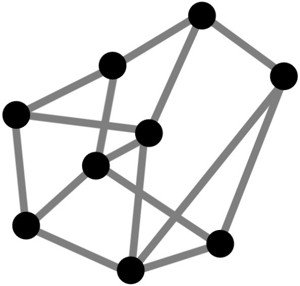

**Theorem (Uses a theorem of Kwan, Letzter, Sudakov, Tran)**

*A class is* **degree-bounded** $\Leftrightarrow$ *it does not contain graphs of arbitrarily large average degree and* **girth** $\geq 6$*.*

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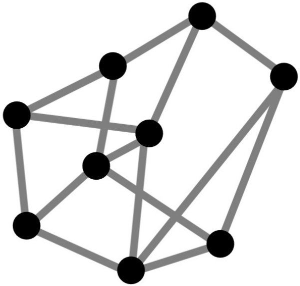

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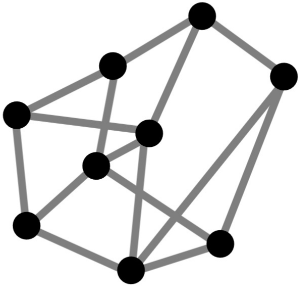

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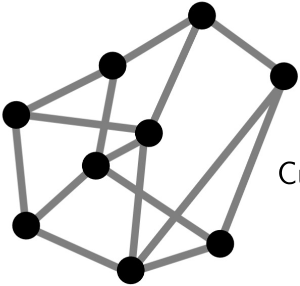

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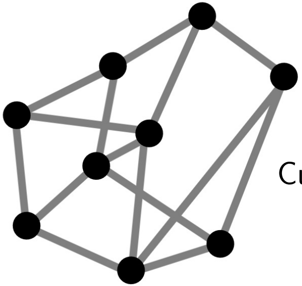

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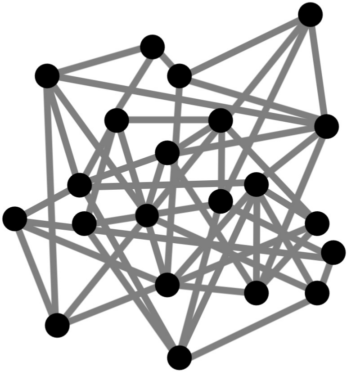
Motivated by problems on the **chromatic number**...

For which classes of graphs does there exist a function $f$ so that $\chi(G) \leq f(\omega(G))$?



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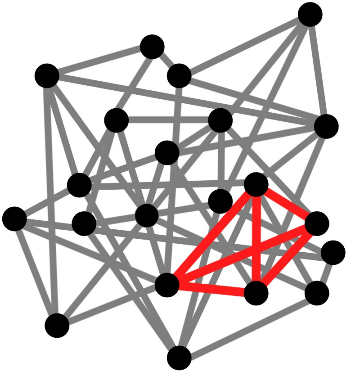

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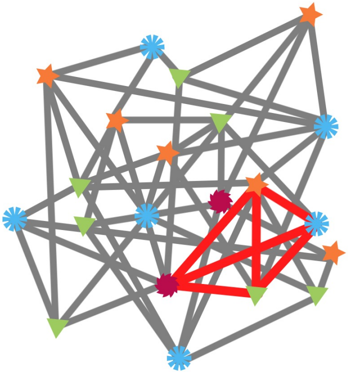

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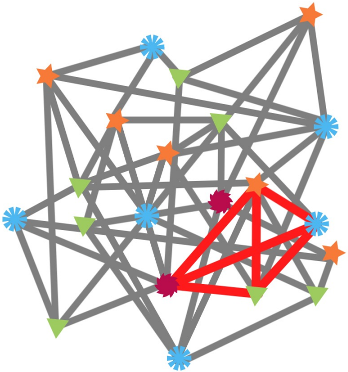

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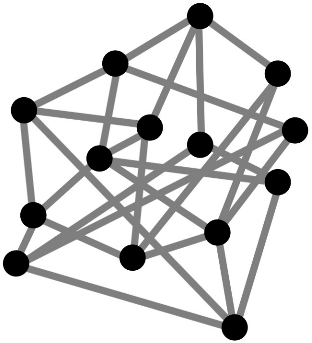

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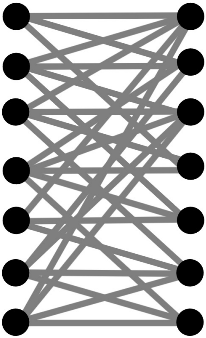

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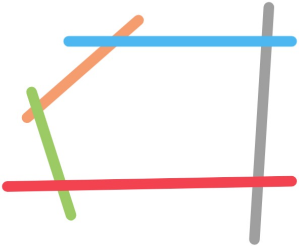

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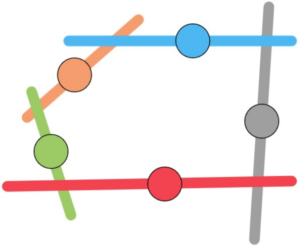

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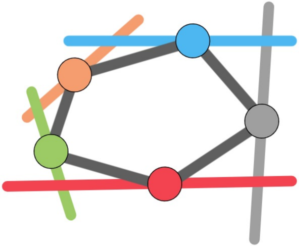

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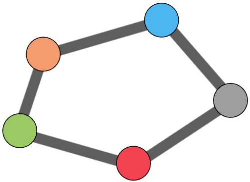

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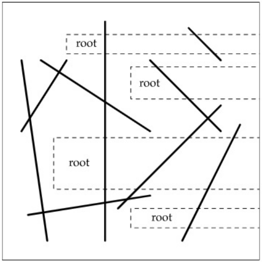


Fig. 1. Segments, probes and roots.

Pawlik, Kozik, Krawczyk Lasoń, Micek, Trotter, & Walczak showed that they are not $\chi$-bounded.

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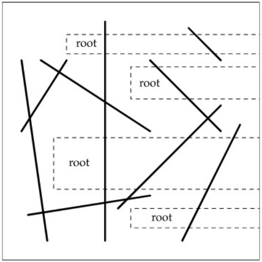


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All of their induced subgraphs with **girth** $\geq 5$ have bounded average degree.

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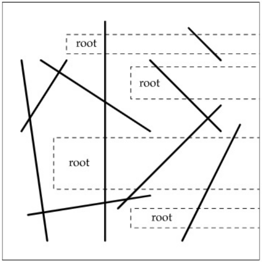


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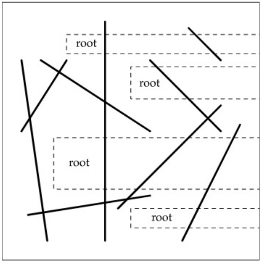


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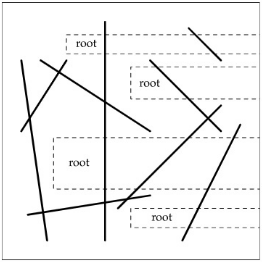


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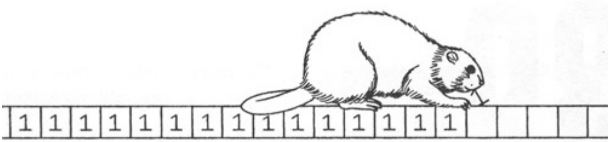
No general bound, not even $2 \uparrow^{\omega} \omega$ or...




(Image from "The New Turing Omnibus")

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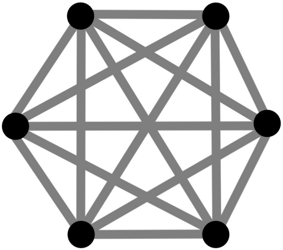
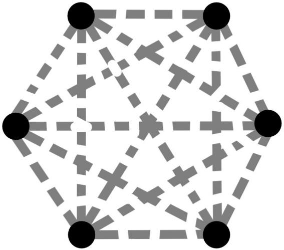


$\geq n^{\epsilon}$



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Scott, Seymour, Spirkl: $\chi \leq \omega^{\log_2(\omega)}$

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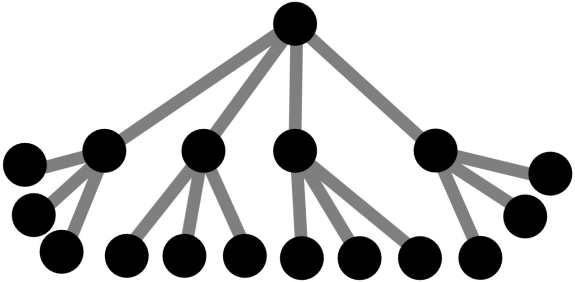
*For any forest $F$, the class of graphs with no induced $F$ has avgdeg $\leq$ **poly**$(\tau)$.*



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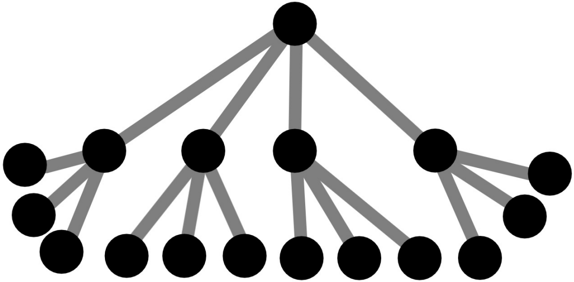

It is not known if these classes are $\chi$-**bounded**;
this is the Gyárfás–Sumner Conjecture.

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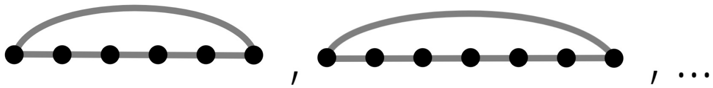

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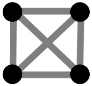

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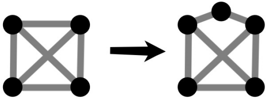

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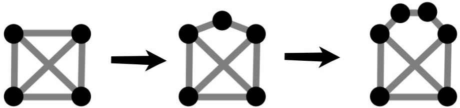

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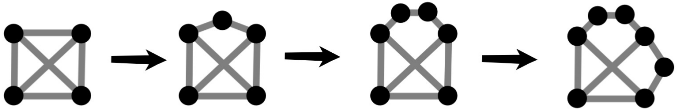
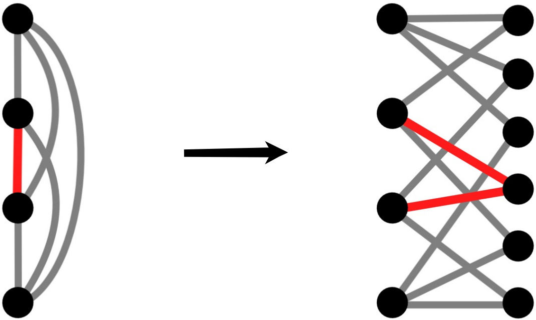


The class of 2-degenerate graphs contains all graphs as **induced subdivisions**.



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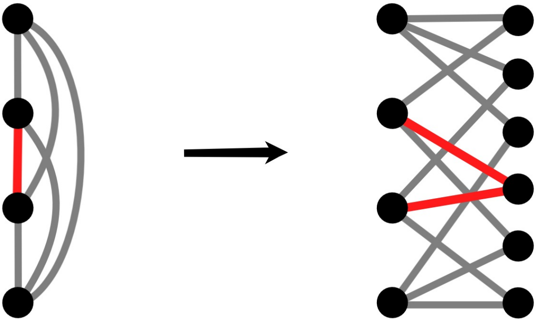

For any $d$, the class of $d$-degenerate graphs is **degree-bounded**.

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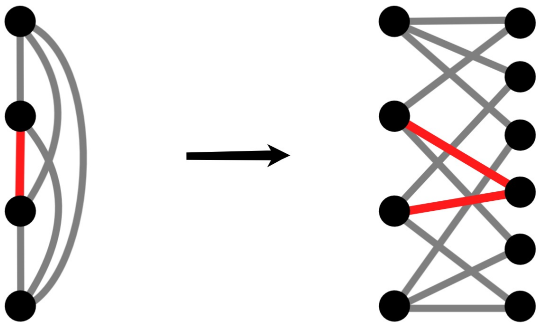
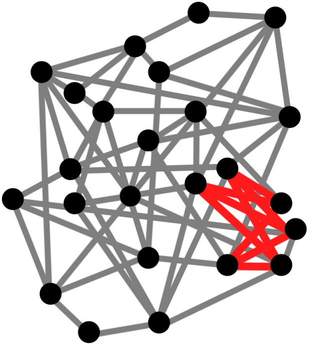


Theorem (Uses a theorem of Kwan, Letzter, Sudakov, Tran)

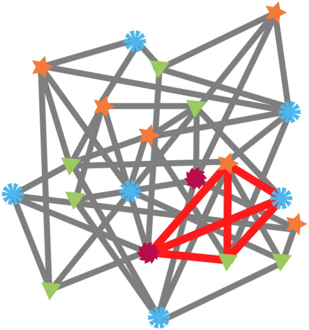
*A class is* **degree-bounded** $\Leftrightarrow$ *its graphs of* **girth** $\geq 6$ *have bounded degeneracy.*

**degree-bounded:**



$$\tau \leq \text{avgdeg} \leq f(\tau)$$

**$\chi$-bounded:**



$$\omega \leq \chi \leq f(\omega)$$

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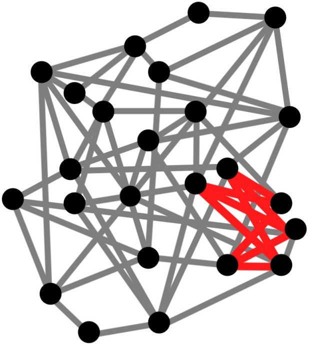

$\tau \leq \text{avgdeg} \leq 2^{2^{2^{2^{poly(\tau)}}}}$

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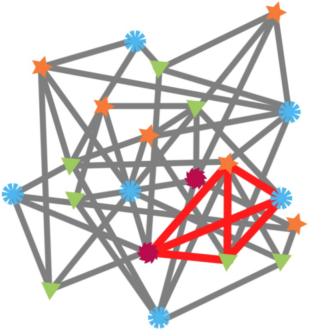

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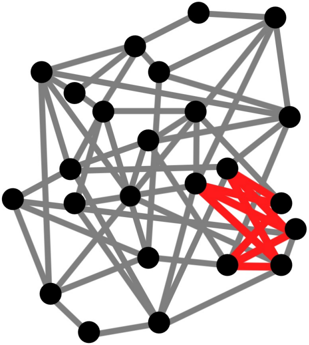

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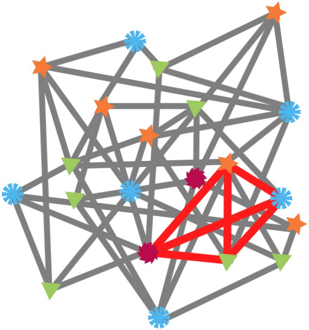

$\omega \leq \chi \leq f(\omega)$

Maybe both $\leq$ polynomial.

# Pause.

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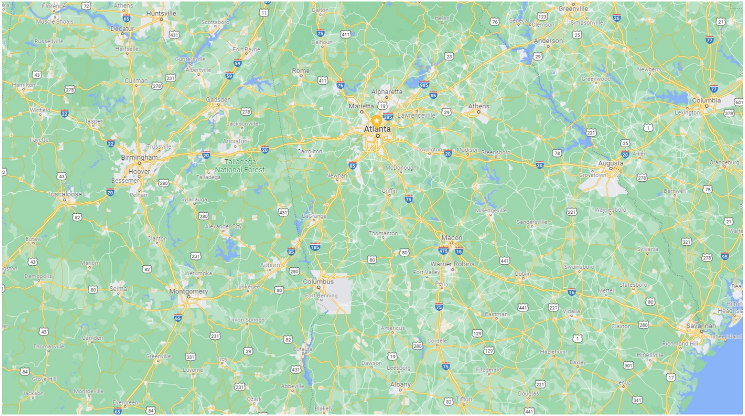

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Fort Payne
Calhoun
Albertville
Rome
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Alpharetta
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Lawrenceville
Athens
Greenwood
Newberry
Columbia
Lexington
Winfield
Jasper
Jacksonville
Atlanta
Anniston
Trussville
Fayette
Birmingham
Hoover
Bessemer
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National Forest
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Carrollton
Covington
Madison
Greensboro
Augusta
Aiken
Grovetown
Orangeburg
Newnan
McDonough
Tuscaloosa
Pelham
Sylacauga
Griffin
Milledgeville
Sandersville
Waynesboro
Barnwell
LaGrange
Alexander City
Thomaston
Macon
Clanton
Eutaw
Warner Robins
Sylvania
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Fort Valley
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Swainsboro
Marion
Demopolis
Wetumpka
Columbus
Perry
Statesboro
Selma
Montgomery
Tuskegee
Fort Benning
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Union Springs
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Vidalia
Claxton
Savannah
Americus
Cordele
Camden
Thomasville
Greenville
Troy
Eufaula
Hazlehurst
Baxley
Richmond Hill
Hinesville
Tybee Island
Dawson
Leesburg
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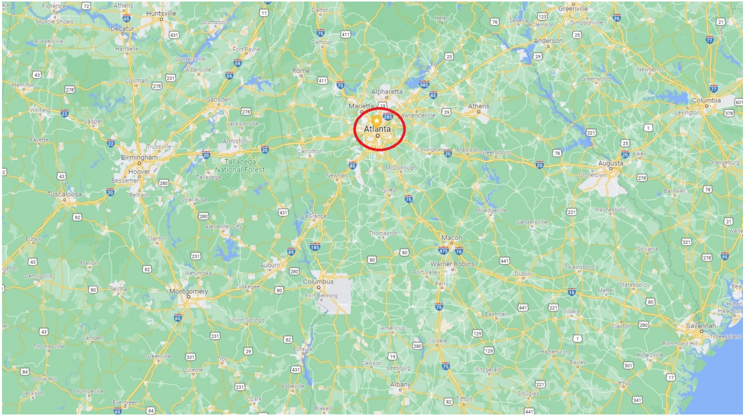

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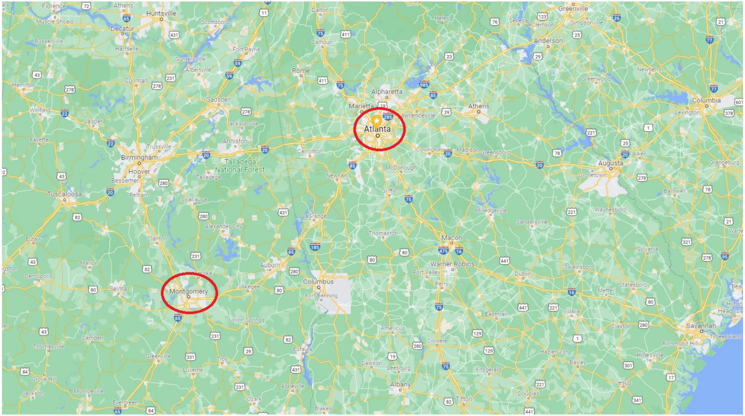

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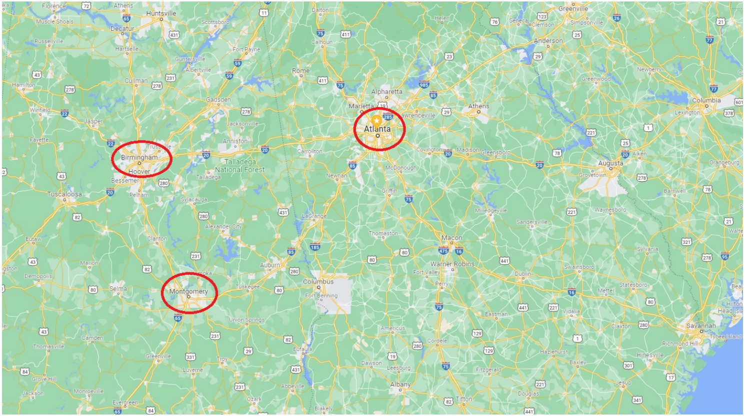

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# Obtaining/improving the bound $\mathrm{avgdeg} \leq 2^{2^{2^{2^{poly(\tau)}}}}$

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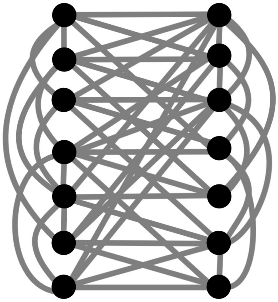
**Proposition**

*For any $d$, every graph of avgdeg $\geq 2d$ has a* **bipartite** *subgraph with avgdeg $\geq d$.*



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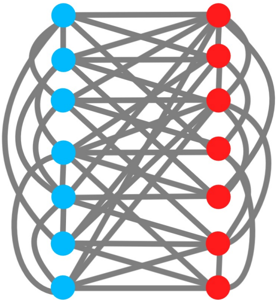

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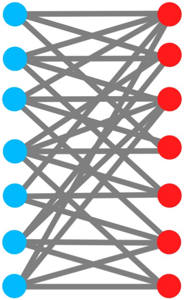

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**Theorem (Kwan, Letzter, Sudakov, & Tran, 2020)**

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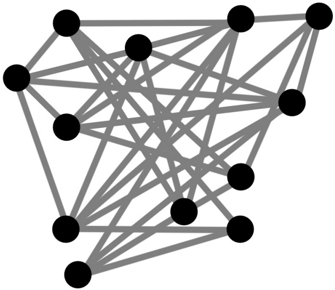

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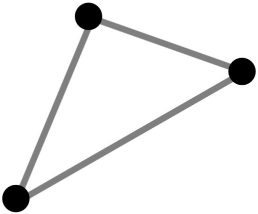

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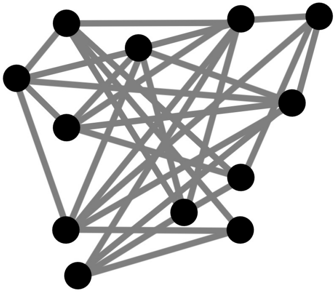

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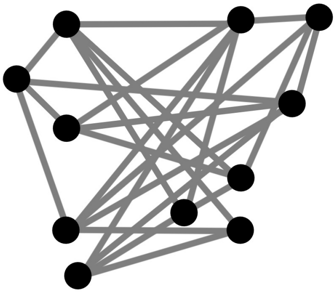

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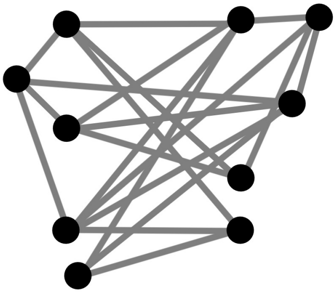

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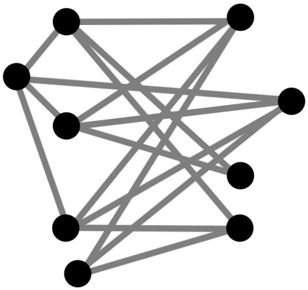

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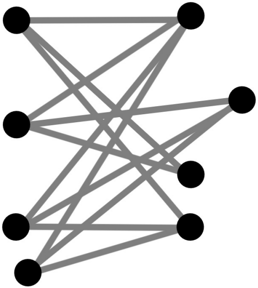

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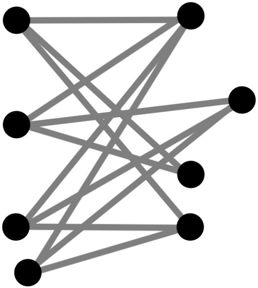

The function must be exponential in $d$.

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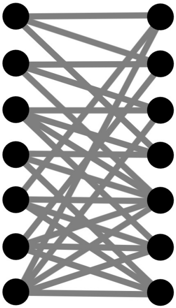
**Theorem (Kühn & Osthus, 04)**

*For any $d$, every bipartite graph of avgdeg $\ge 2^{2^{poly(d)}}$ has a subgraph with avgdeg $\ge d$ and* **no 4-cycles**.



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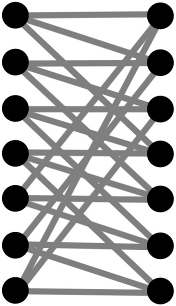

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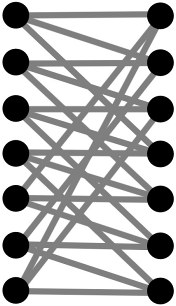

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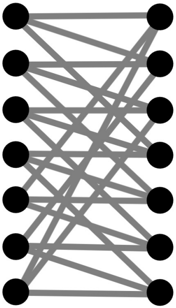

Showed a lower bound of $d^{3-o(1)}$.

# Obtaining/improving the bound $\mathrm{avgdeg} \le 2^{2^{2^{2^{poly(\tau)}}}}$

**Theorem**

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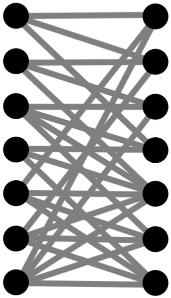

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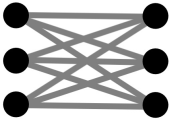

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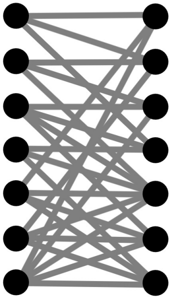

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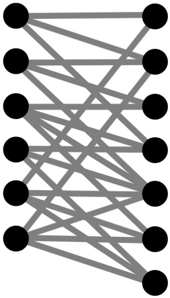

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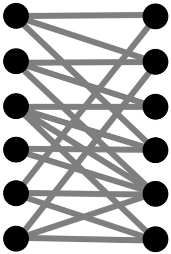

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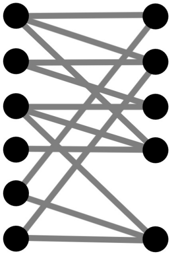

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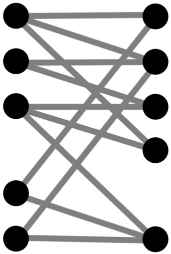

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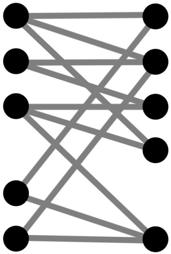
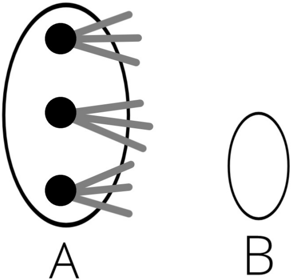


Based on a proof of Dellamonica, Koubek, Martin, & Rödl, 11.

# Obtaining/improving the bound $\mathrm{avgdeg} \leq 2^{2^{2^{2^{poly(\tau)}}}}$

**Lemma**

*For any $r, \lambda \geq 1$, every bipartite graph of avgdeg $\geq f(r, \lambda)$ has an* **induced** *$(r, \lambda)$-subgraph.*



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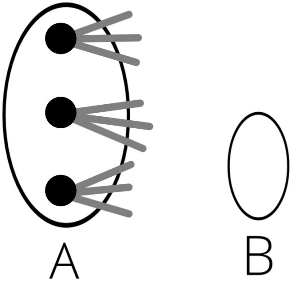


This lets us apply (Füredi, 83).

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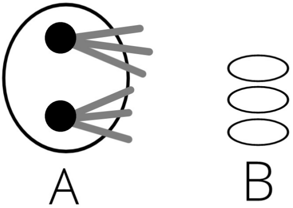


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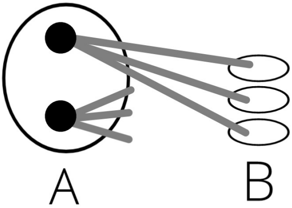


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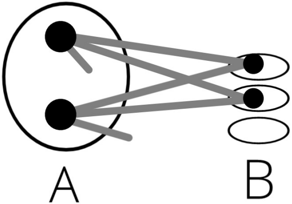


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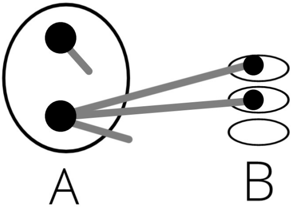


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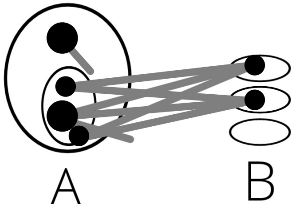


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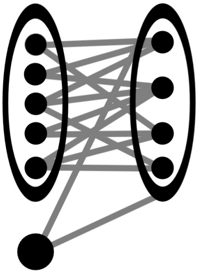

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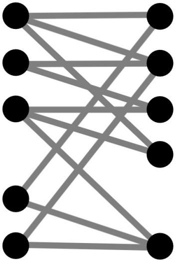

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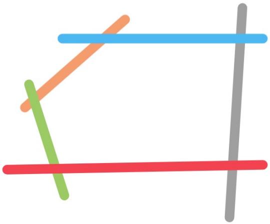

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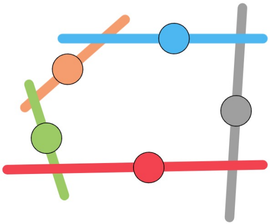

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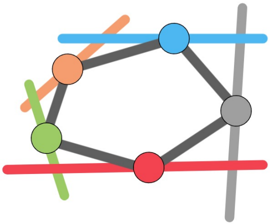

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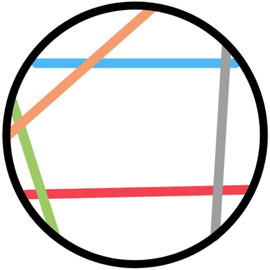

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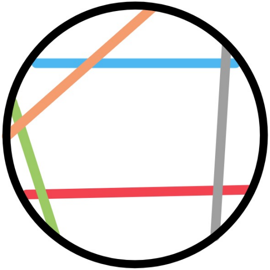

This class is $\chi$-bounded w/ **polynomial** bound (w/ Davies).

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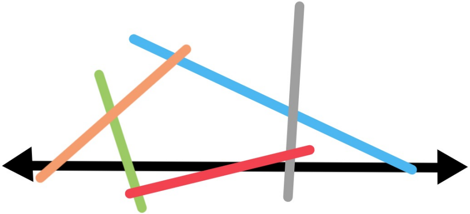

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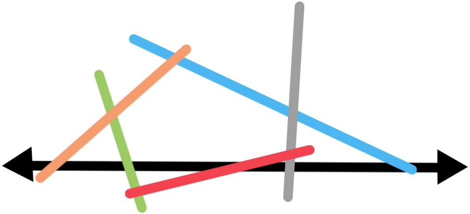

This class is $\chi$-bounded, but is there a **polynomial** bound?

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Are these conjectures consistent?

**Thank you!**