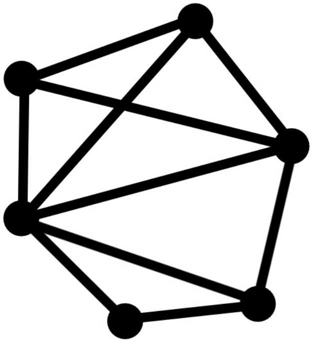
Colouring Graphs with Geometric Origins

Rose McCarty

September 2019

A **graph** is a mathematical formalization of a network consisting of nodes and edges.



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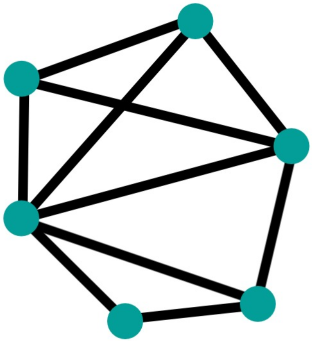

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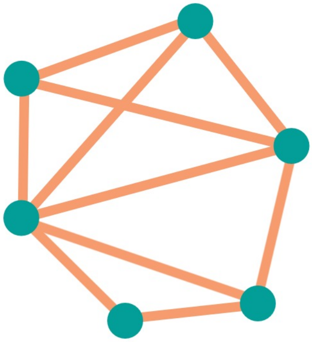
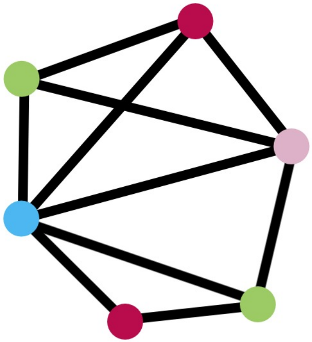


$$G = (V, E)$$

Graphs arise from a variety of contexts:

- The internet, social networks, distribution of goods, circuit design, job applications, UML diagrams, . . .
- Permutations, groups, posets, knots, polyhedra, CW complexes, . . .

We are interested in colourings of the nodes where no edge joins two nodes of the same colour.



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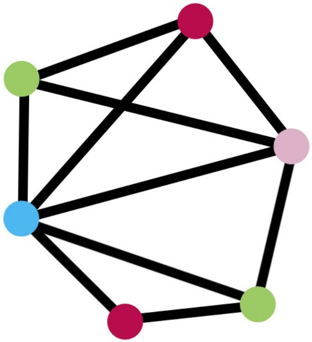

The **chromatic number** $\chi(G)$ is the minimum number of colours in such a colouring.

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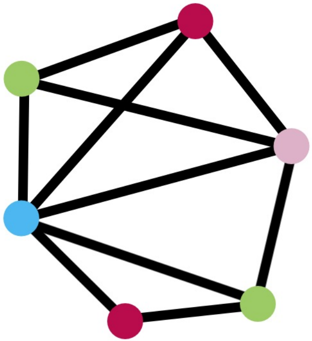

To show that $\chi(G) \leq k$, just give a good colouring.

How do you show that $\chi(G) \geq k$?

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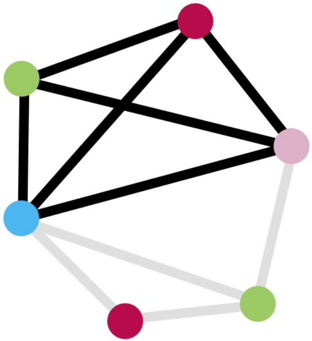

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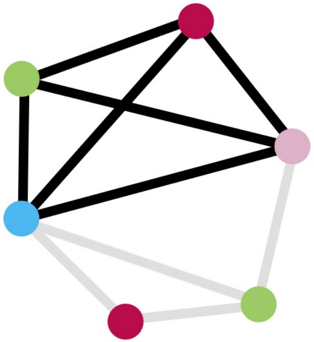
A **clique** is a set of pairwise adjacent nodes.



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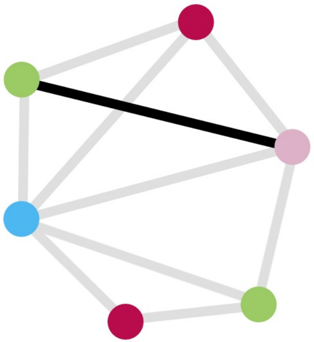

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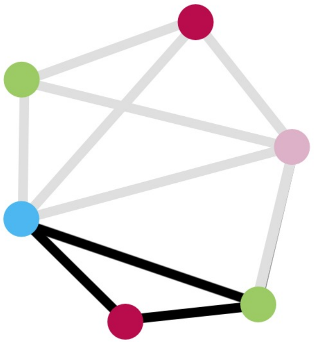

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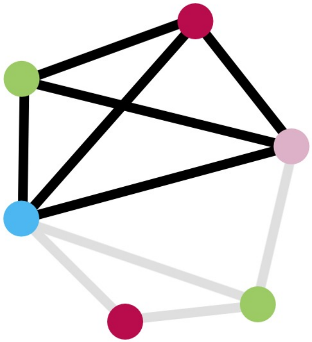

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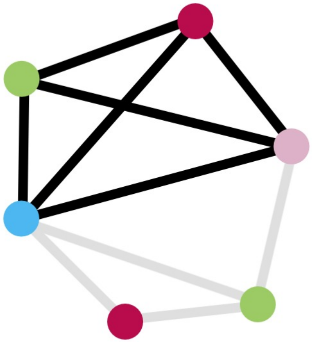
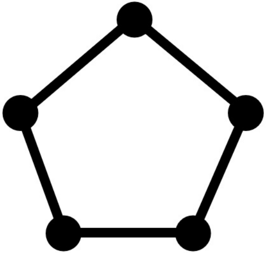


The **clique number** $\omega(G)$ is the largest size of a clique.

$$\chi(G) \geq \omega(G)$$

This graph has clique number 2.



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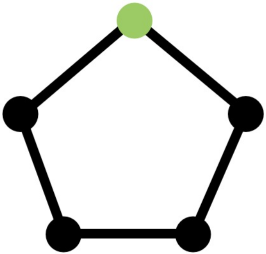

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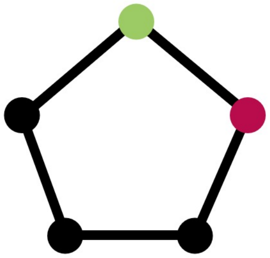

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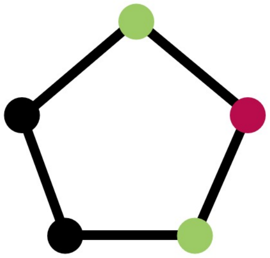

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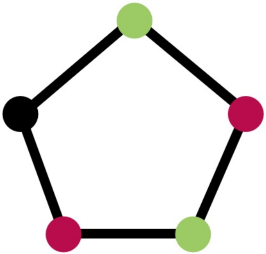

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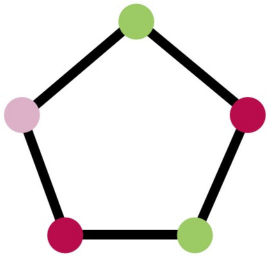

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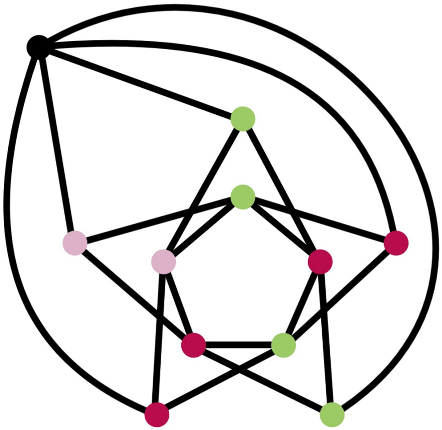
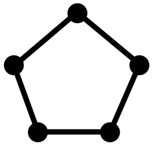
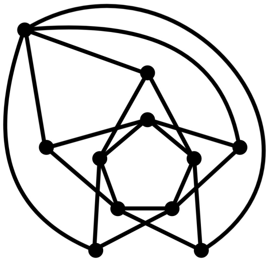


but chromatic number 4.

Theorem (Mycielski 1955)

For every integer k, there exists a graph G with $\omega(G) = 2$ and $\chi(G) \geq k$.





...

For what classes of graphs is χ upper-bounded by a function of ω?

$$\chi \leq \omega$$

$$\chi \leq \omega^3$$

$$\chi \leq 2^\omega$$

$$\chi \leq \omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}}$$

$$\cdots$$

Such classes are called χ-**bounded.**

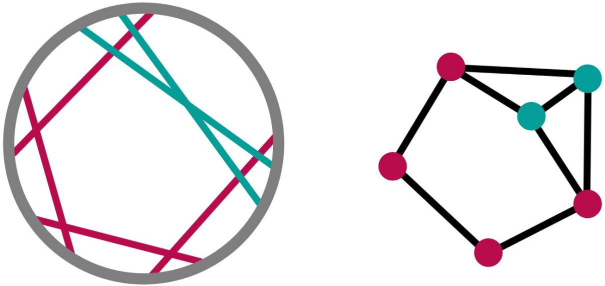
A class is **hereditary** if it is closed under deleting vertices.

Conjecture (Esperet 2012)

For every hereditary class of graphs with $\chi \leq f(\omega)$, there exists d so that $\chi \leq \omega^d$.

i.e. if $\chi \leq \omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}}$ then $\chi \leq \omega^8$ too!

Let's look at the class of **circle graphs**.



Take a circle with some chords.

Make a node for each chord.

Make two nodes adjacent if their chords intersect.

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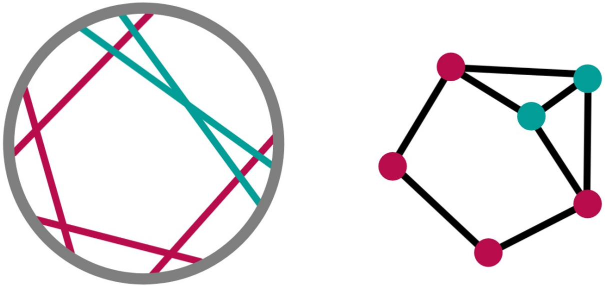

Gyárfás 1985	$\chi \leq \omega^2 \cdot 2^{2\omega}$
Kostochka 1988	$\chi \leq \omega^3 \cdot 2^{\omega}$
Kostochka-Kratochvíl 1997	$\chi \leq 50 \cdot 2^{\omega}$

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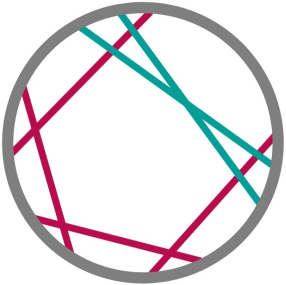
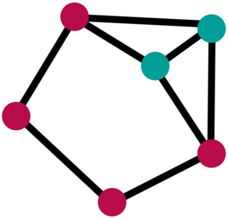


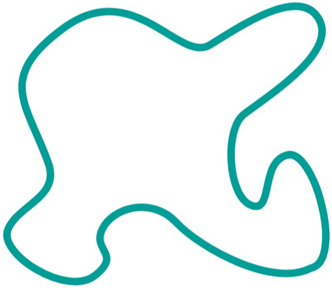


Theorem (Davies-M. 2019)

Every circle graph has $\chi \leq 7\omega^2$.

Bartosz Walczak, James Davies, me, Tomasz Krawczyk

Consider a Jordan curve in the plane.



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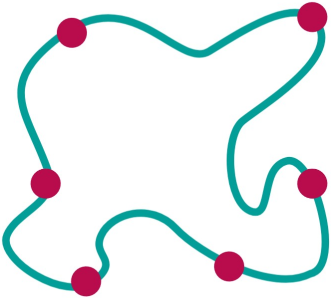

Place points on it, and make a node for each point.

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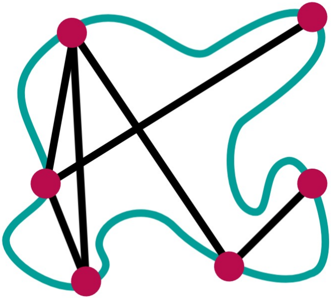

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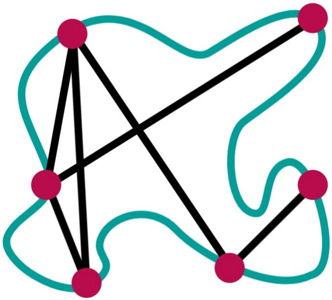

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DKMW 2019+: $\chi \leq 2^{2\omega}$

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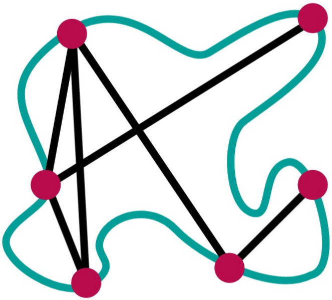

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DKMW: $\chi \leq 2^{2\omega}$ **Question:** $\chi \leq \omega^d$???