Vertex-minors and structure for dense graphs

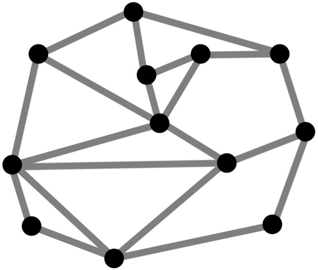
Rose McCarty

Department of Combinatorics and Optimization
University of Waterloo

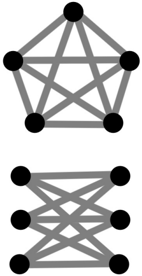
Joint work with Jim Geelen and Paul Wollan

Kuratowski's Theorem

A graph is planar iff it has no K_5 or $K_{3,3}$ minor.



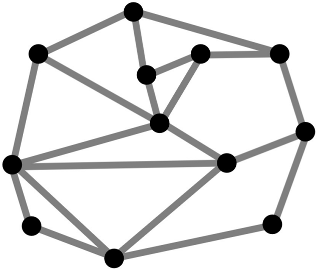
planar graphs



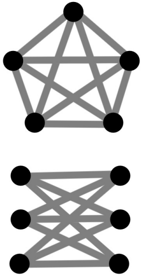
forbidden minors

Theorem (Robertson & Seymour 2004)

Every minor-closed class has **finitely many** *minimal forbidden minors.*



planar graphs

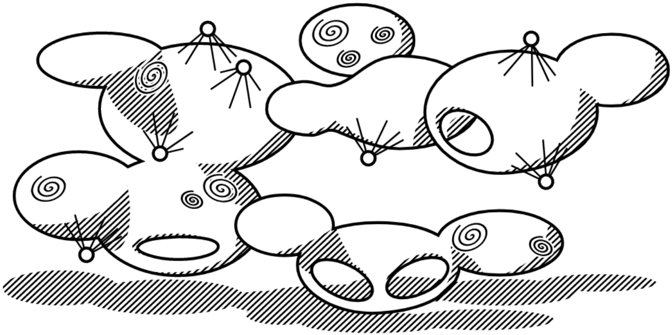


forbidden minors

Theorem (Robertson & Seymour 2003)

The graphs in any proper minor-closed class "almost embed" in a surface of bounded genus.

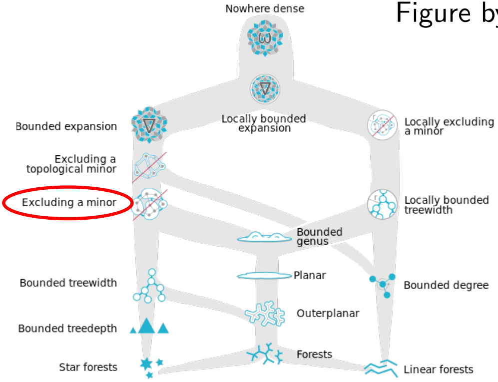
Figure by Felix Reidl



Theorem (Robertson & Seymour 2003)

The graphs in any proper minor-closed class "almost embed" in a surface of bounded genus.

Figure by Felix Reidl

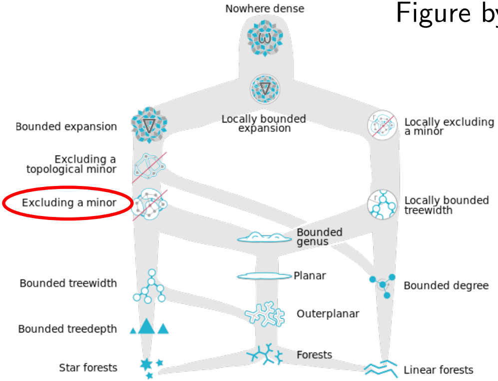


Theory of "**sparsity**" (Nešetřil & Ossona de Mendez)

Theorem (Robertson & Seymour 2003)

The graphs in any proper minor-closed class "almost embed" in a surface of bounded genus.

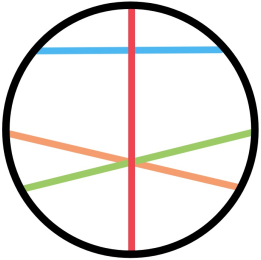
Figure by Felix Reidl



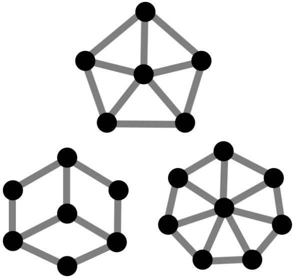
What are the "**dense**" analogs?

Bouchet's Theorem

A graph is a **circle graph** *iff it has no W_5, $\hat{W}_6$, or W_7* **vertex-minor**.



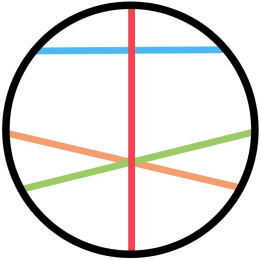
circle graphs



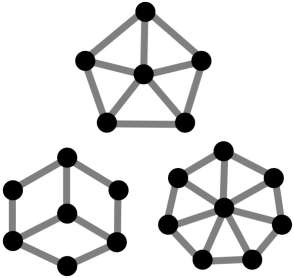
forbidden vertex-minors

Conjecture (Oum 2017)

Every **vertex-minor***-closed class has* **finitely many** *minimal forbidden vertex-minors.*



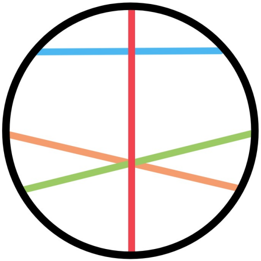
circle graphs



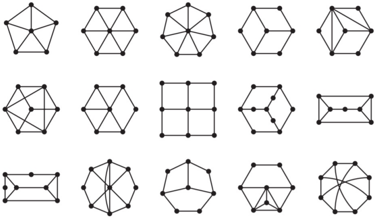
forbidden vertex-minors

Geelen and Oum's Theorem

A graph is a **circle graph** *iff it has no* $W_5, W_6, \ldots$ **pivot-minor**.



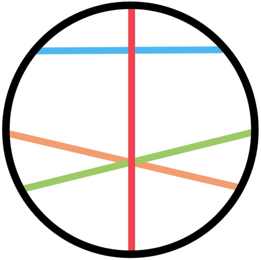
circle graphs



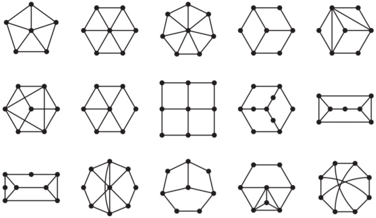
forbidden pivot-minors

Geelen and Oum's Theorem

A graph is a **circle graph** *iff it has no* $W_5, W_6, \ldots$ **pivot-minor**.



circle graphs



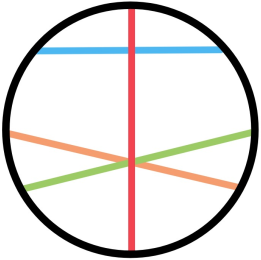
forbidden pivot-minors

Common generalization!

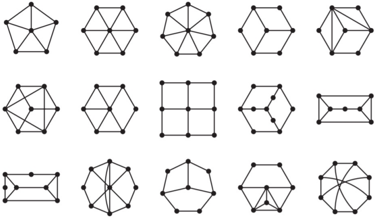
(Bouchet 1988; de Fraysseix 1981)

Conjecture (Oum 2017)

Every **pivot-minor***-closed class has* **finitely many** *minimal forbidden pivot-minors.*



circle graphs



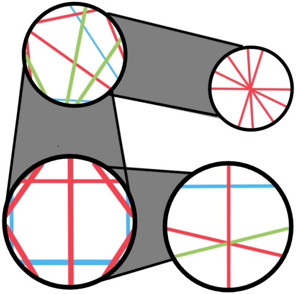
forbidden pivot-minors

Common generalization!

(Bouchet 1988; de Fraysseix 1981)

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*



Ongoing project with Jim Geelen & Paul Wollan.

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

Theorem (Geelen, Kwon, McCarty, Wollan)

A class of graphs "fully decomposes" iff it does not have all **circle graphs** *as* **vertex-minors***.*

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

Theorem (Geelen, Kwon, McCarty, Wollan)

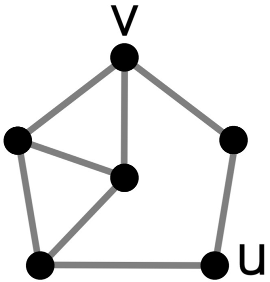
A class of graphs "fully decomposes" iff it does not have all **circle graphs** *as* **vertex-minors***.*

Theorem (Geelen, McCarty, & Wollan)

Relative to a "highly-connected" circle graph, the rest of the graph "almost attaches compatibly".

The **vertex-minors** of a graph G are obtained by

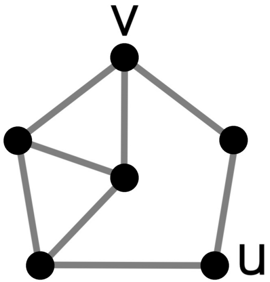
1) vertex deletion and
2) **local complementation**



G

The **vertex-minors** of a graph G are obtained by

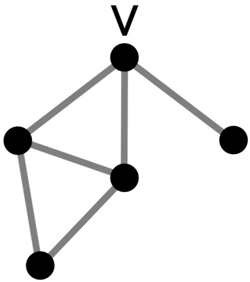
1) vertex deletion and
2) **local complementation**



G

The **vertex-minors** of a graph G are obtained by

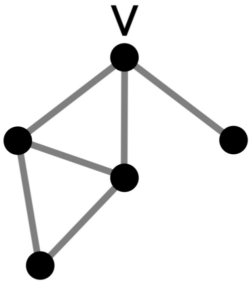
1) vertex deletion and
2) **local complementation**



$G - u$

The **vertex-minors** of a graph G are obtained by

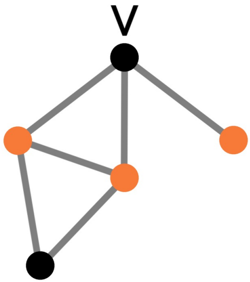
1) vertex deletion and
2) **local complementation**



$G - u$

The **vertex-minors** of a graph G are obtained by

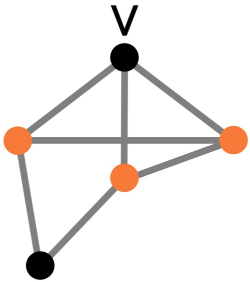
1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



$$(G - u)$$

The **vertex-minors** of a graph G are obtained by

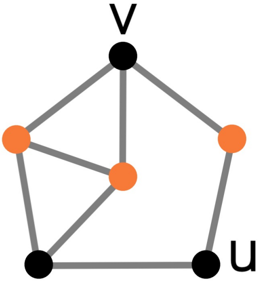
1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



$$(G - u) * v$$

The **vertex-minors** of a graph G are obtained by

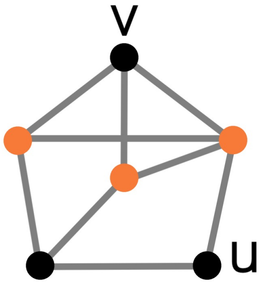
1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



G

The **vertex-minors** of a graph G are obtained by

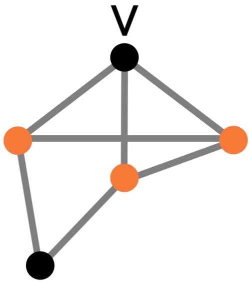
1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



$G * v$

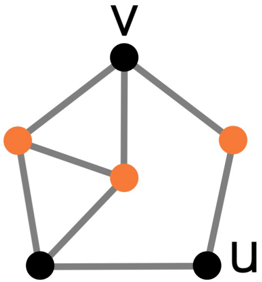
The **vertex-minors** of a graph G are obtained by

1) vertex deletion and
2) **local complementation** (replace the induced subgraph on the neighborhood of v by its complement).



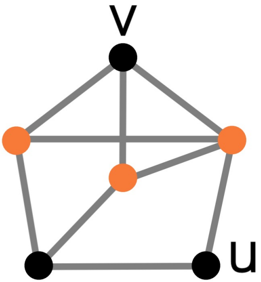
$$G * v - u$$

The **vertex-minors** of a graph G are the induced subgraphs of graphs that are **locally equivalent** to G.



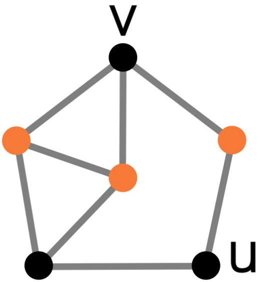
G

The **vertex-minors** of a graph G are the induced subgraphs of graphs that are **locally equivalent** to G.



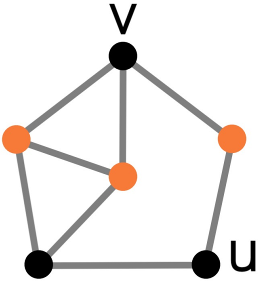
$G * v$

The **vertex-minors** of a graph G are the induced subgraphs of graphs that are **locally equivalent** to G.



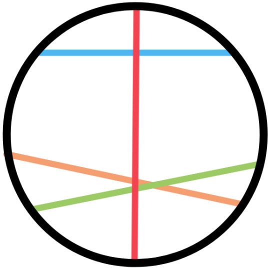
$$G * v * v$$

The **vertex-minors** of a graph G are the induced subgraphs of graphs that are **locally equivalent** to G.

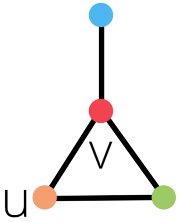


$$G * v * v = G$$

A **circle graph** is the intersection graph of chords on a circle.
Circle graphs are closed under local complementation and vertex-deletion.

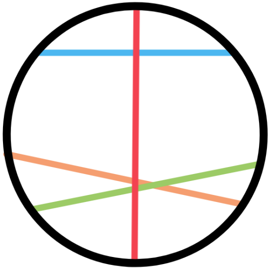


chord diagram

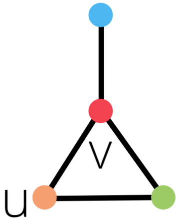


circle graph G

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.

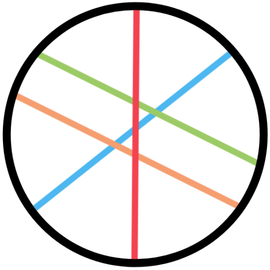


chord diagram

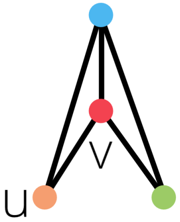


circle graph G

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.

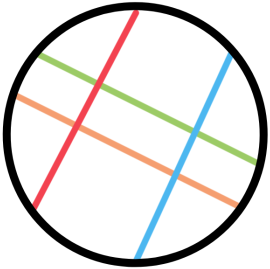


chord diagram

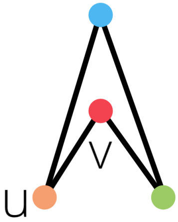


circle graph $G * v$

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.

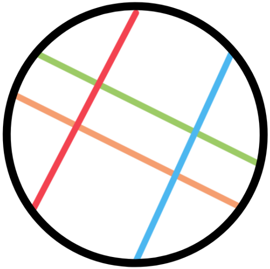


chord diagram

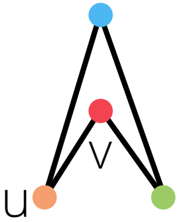


circle graph $G * v * u$

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.

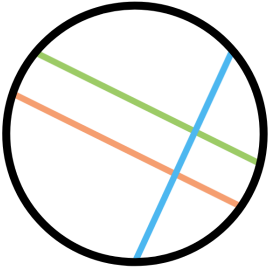


chord diagram

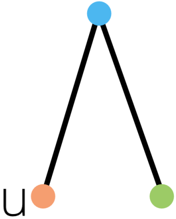


circle graph $G * v * u$

A **circle graph** is the intersection graph of chords on a circle. Circle graphs are closed under local complementation and vertex-deletion.



chord diagram



circle graph $G * v * u - v$

Why **local equivalence** classes?

- graph states in quantum computing

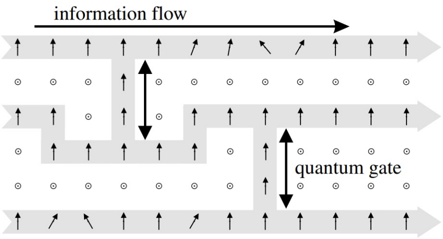


FIG. 1. Quantum computation by measuring two-state parti-

(Raussendorf-Briegel, Van den Nest-Dehaene-De Moor)

Why **local equivalence** classes?

- graph states in quantum computing

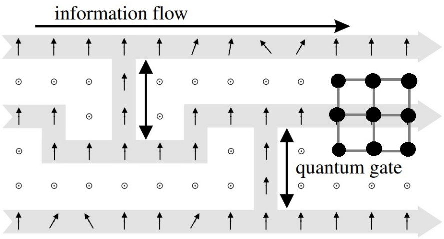


FIG. 1. Quantum computation by measuring two-state parti-

(Raussendorf-Briegel, Van den Nest-Dehaene-De Moor)

Why **local equivalence** classes?

- graph states in quantum computing

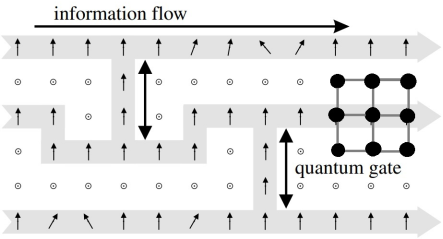


FIG. 1. Quantum computation by measuring two-state parti-

(Raussendorf-Briegel, Van den Nest-Dehaene-De Moor)

Conjecture (Geelen)

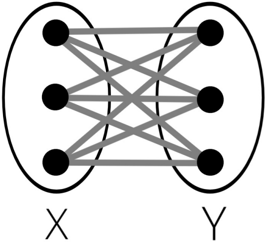
When the graph states that can be prepared have no ***H-vertex-minor****,* $BQP_H = BPP$.

Why **local equivalence** classes?

- graph states in quantum computing
- **rank-connectivity**

$$
\begin{array}{c c}
 & \begin{array}{cccccc} & X & & & Y & \end{array} \\
\begin{array}{c} \\ X \\ \\ \\ Y \\ \\ \end{array} &
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

adjacency matrix

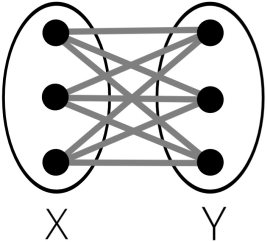


Why **local equivalence** classes?

- graph states in quantum computing
- **rank-connectivity**

$$\begin{array}{cc} & \begin{matrix} \quad & X & \quad & \quad & Y & \quad \end{matrix} \\ \begin{matrix} \\ X \\ \\ \\ Y \\ \\ \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

adjacency matrix

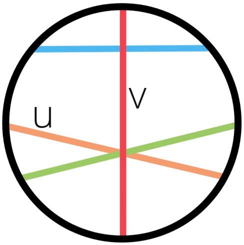


Conjecture (Geelen)

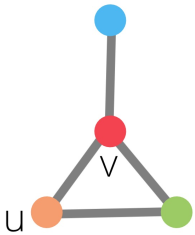
The graphs in any proper **vertex-minor***-closed class "***decompose***" into parts that are "almost"* **circle graphs***.*

Why **local equivalence** classes?

- graph states in quantum computing
- **rank-connectivity**
- has a nice interpretation for **circle graphs**...

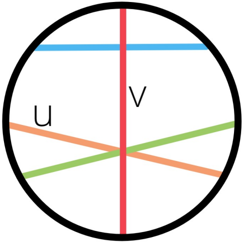


$\longrightarrow$

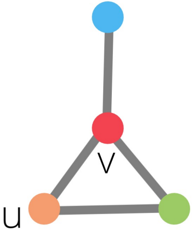


chord diagram

circle graph

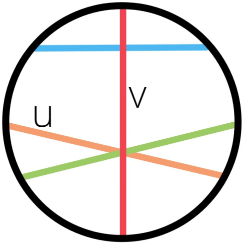


chord diagram

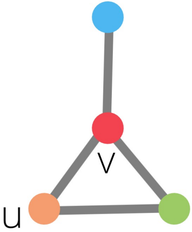


circle graph

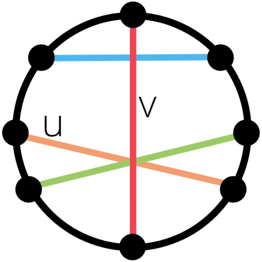
tour graph



chord diagram

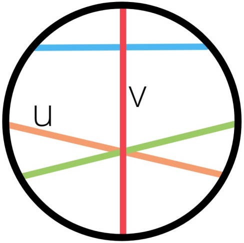


circle graph

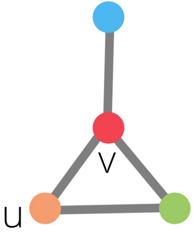


tour graph

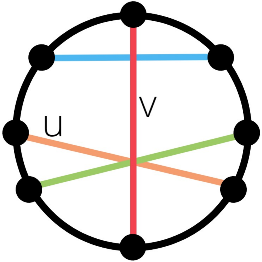
View the **chord diagram** as a 3-regular graph...



chord diagram

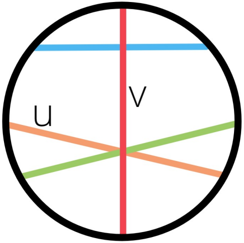


circle graph

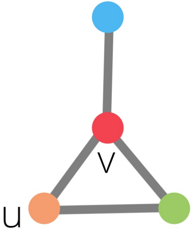


tour graph

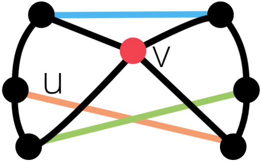
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



chord diagram

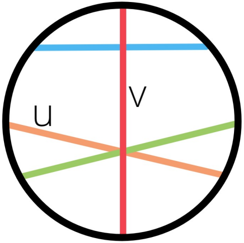


circle graph

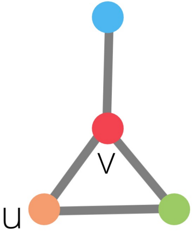


tour graph

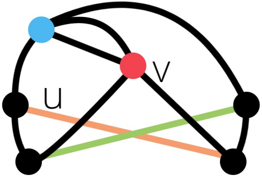
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



chord diagram

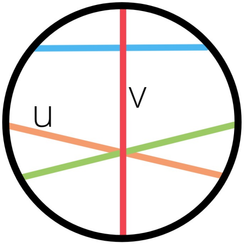


circle graph

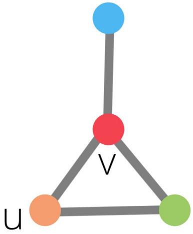


tour graph

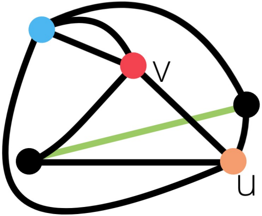
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



chord diagram

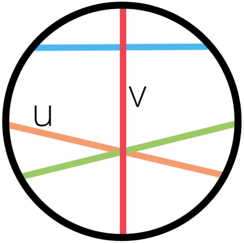


circle graph

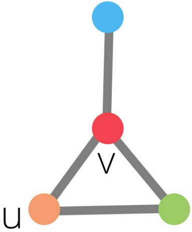


tour graph

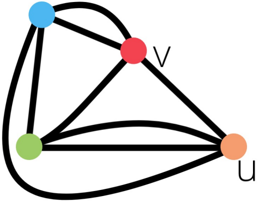
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



chord diagram

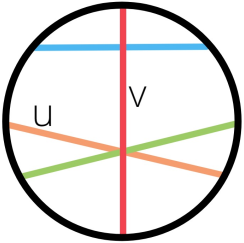


circle graph

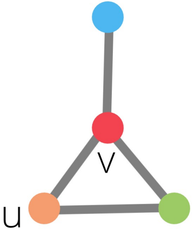


tour graph

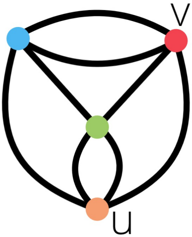
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



chord diagram

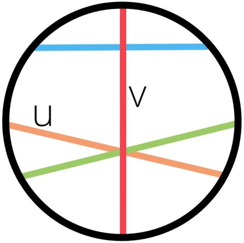


circle graph

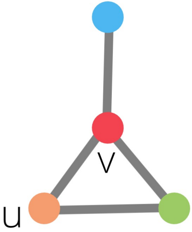


tour graph

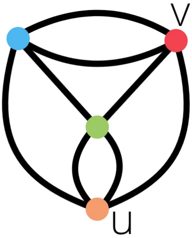
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**.



chord diagram

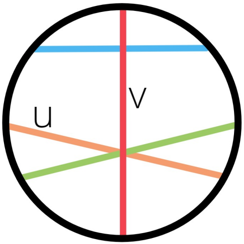


circle graph

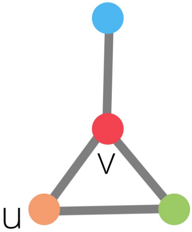


tour graph

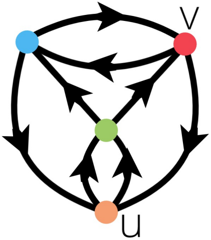
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit.



chord diagram

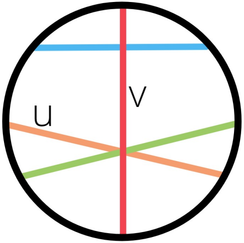


circle graph

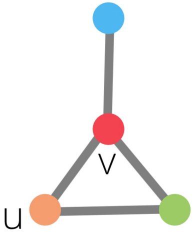


tour graph

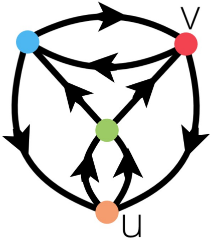
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit.



chord diagram

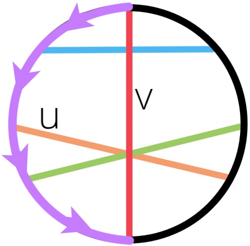


circle graph

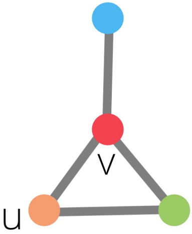


tour graph

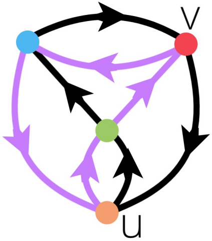
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v.



chord diagram

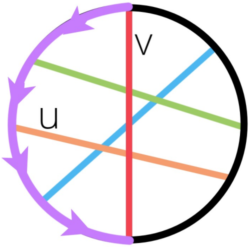


circle graph

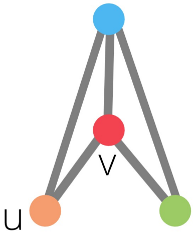


tour graph

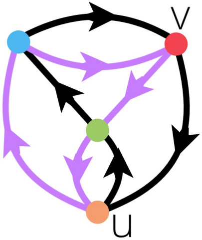
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v.



chord diagram

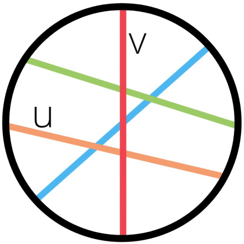


circle graph

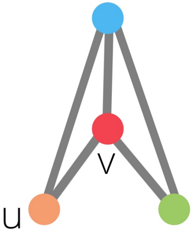


tour graph

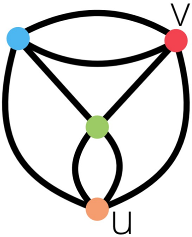
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v.



chord diagram

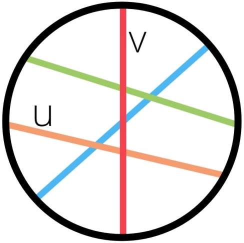


circle graph

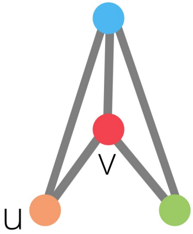


tour graph

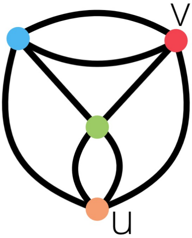
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v.



chord diagram

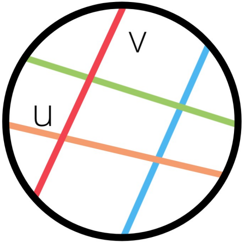


circle graph

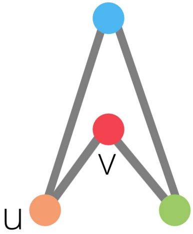


tour graph

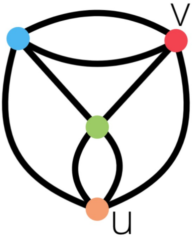
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u.



chord diagram

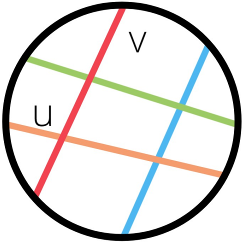


circle graph

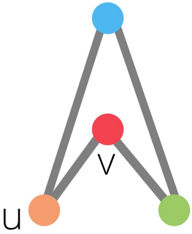


tour graph

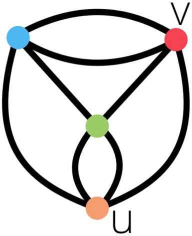
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u.



chord diagram

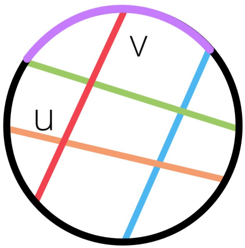


circle graph

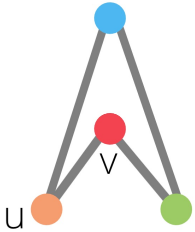


tour graph

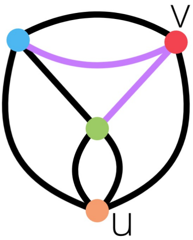
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u. To delete v...



chord diagram

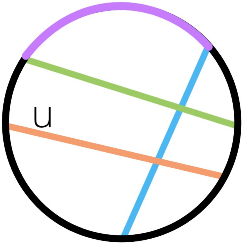


circle graph

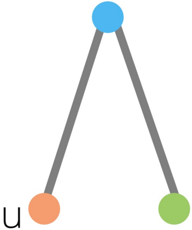


tour graph

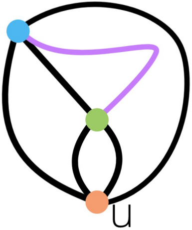
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u. To delete v...



chord diagram

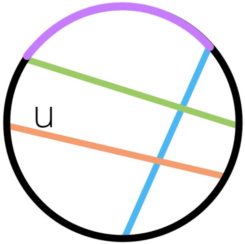


circle graph

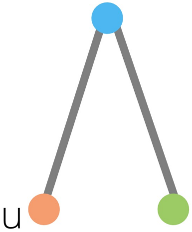


tour graph

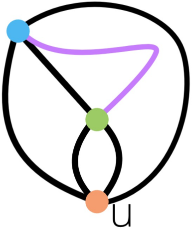
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u. To delete v...



chord diagram

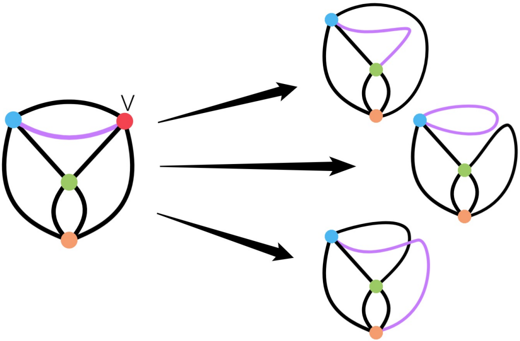


circle graph

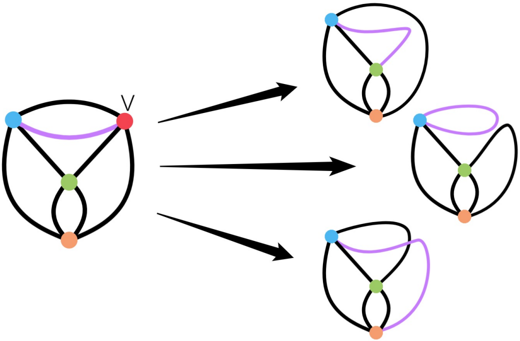


tour graph

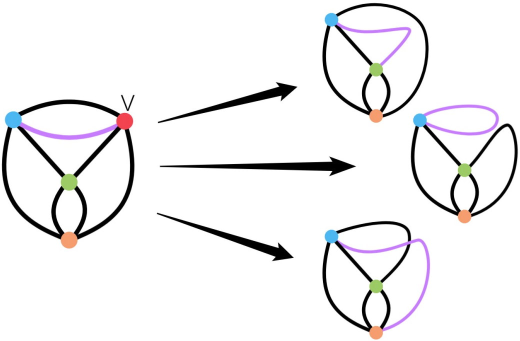
View the **chord diagram** as a 3-regular graph and contract each of the chords to get the **tour graph**. It has a specified Eulerian circuit. Consider locally complementing at v then u. To delete v, **split it off** in the **tour graph**.



In a 4-regular graph, there are 3 ways to **split off** v.



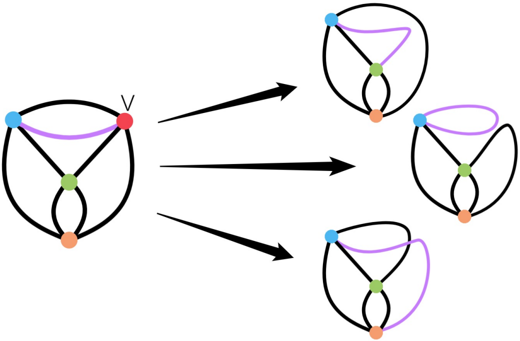
In a 4-regular graph, there are 3 ways to **split off** v. This is how we define **immersions**.



In a 4-regular graph, there are 3 ways to **split off** v. This is how we define **immersions**.

Theorem (Kotzig, Bouchet)

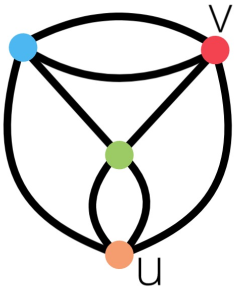
*If H and G are 2-**rank-connected** circle graphs, then H is a vertex-minor of G $\iff$ **tour**(H) **immerses** into **tour**(G).*



Lemma (Bouchet)

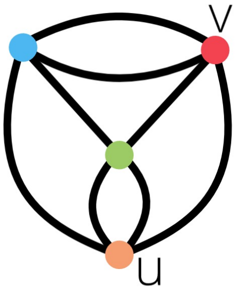
If H is a vertex-minor of G and $v \in V(G) \setminus V(H)$, then H is a **vertex-minor** *of either $G - v$, $G * v - v$, or $G * v * u * v - v$ for each neighbour u of v.*

If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

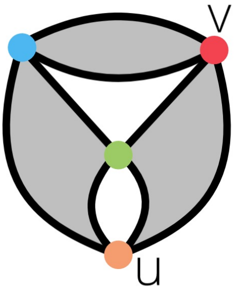
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring.

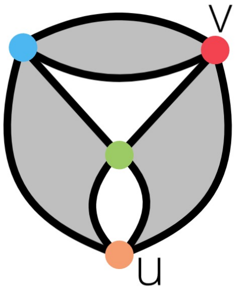
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring.

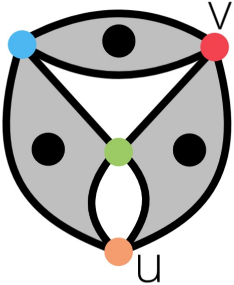
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face,

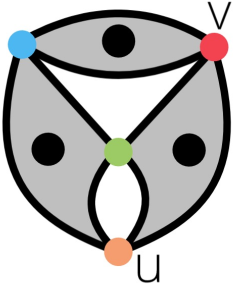
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face,

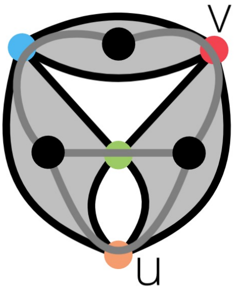
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces.

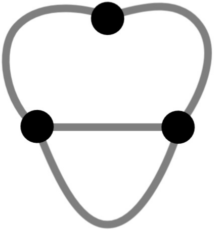
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces.

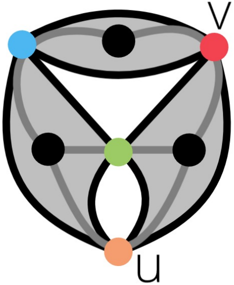
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces.

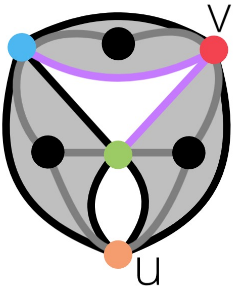
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v.

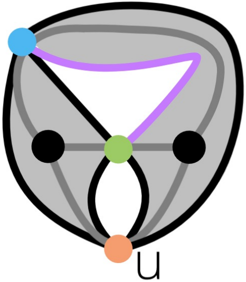
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v.

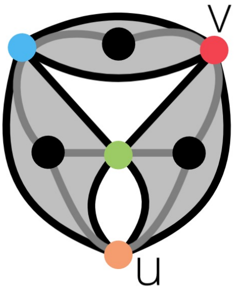
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v.

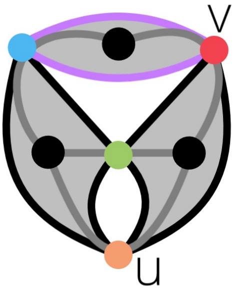
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v.

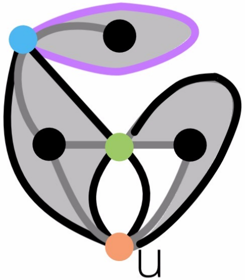
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v.

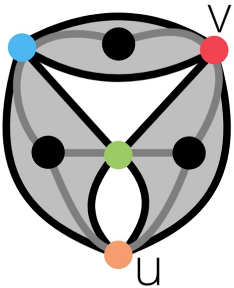
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v.

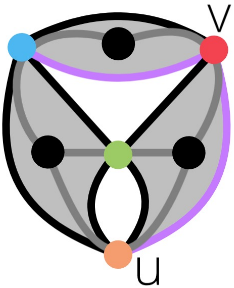
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v. The split that we do not allow...

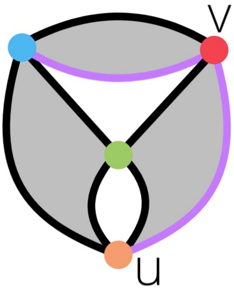
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v. The split that we do not allow...

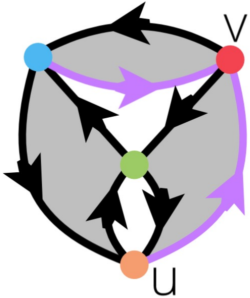
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v. The split that we do not allow...

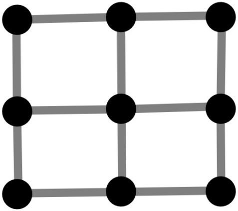
If we only allow 2/3 splits, then this generalizes minors of planar graphs (up to duality).



tour graph

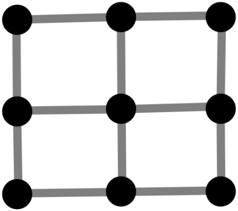
Consider a 2-face-coloring. Put a vertex in each black face, and an edge between touching faces. Consider a split at v. The split that we do not allow breaks the orientation.

Here is the other direction.



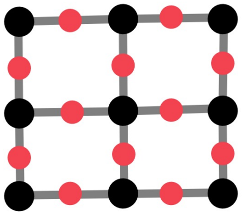
Take a planar graph.

Here is the other direction.



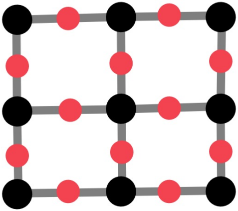
Take a planar graph. Add a vertex to each edge

Here is the other direction.



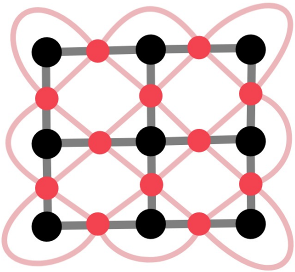
Take a planar graph. Add a vertex to each edge

Here is the other direction.



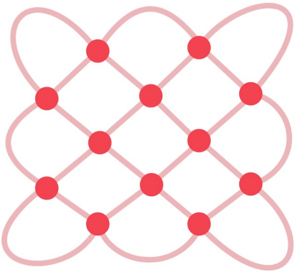
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face.

Here is the other direction.



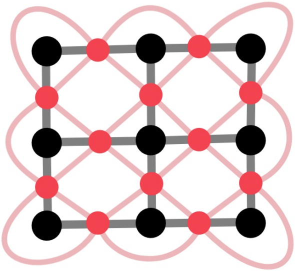
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face.

Here is the other direction.



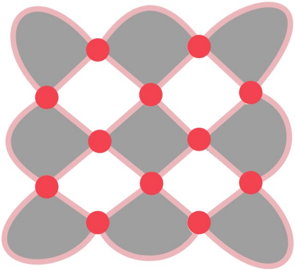
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**.

Here is the other direction.



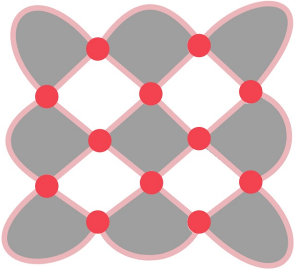
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring.

Here is the other direction.



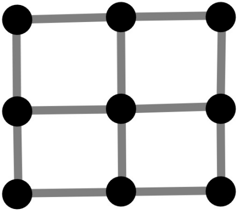
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring.

Here is the other direction.



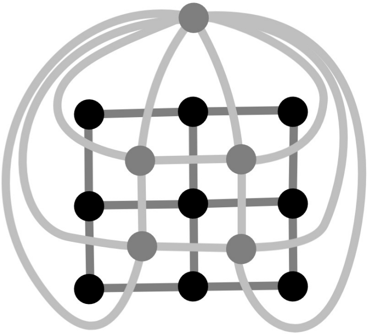
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring. Choosing the other color gives the planar dual.

Here is the other direction.



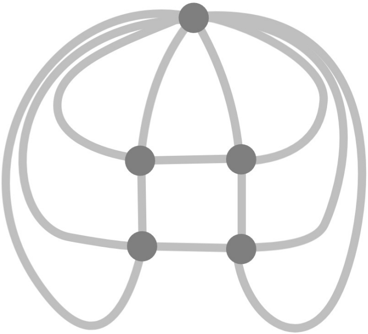
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring. Choosing the other color gives the planar dual.

Here is the other direction.



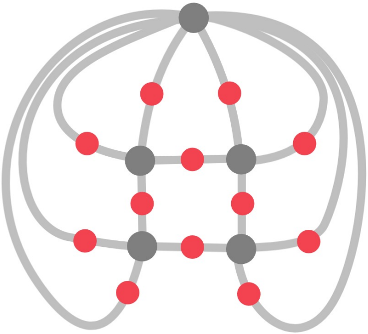
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring. Choosing the other color gives the planar dual.

Here is the other direction.



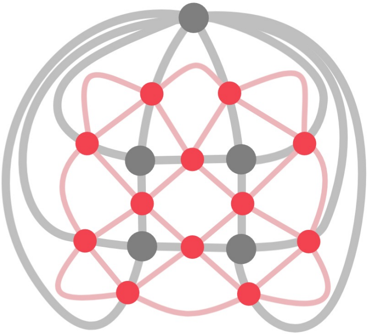
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring. Choosing the other color gives the planar dual.

Here is the other direction.



Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring. Choosing the other color gives the planar dual.

Here is the other direction.

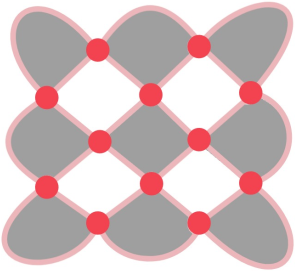


Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring. Choosing the other color gives the planar dual.

Here is the other direction.

Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring. Choosing the other color gives the planar dual.

Here is the other direction.



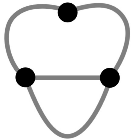
Take a planar graph. Add a vertex to each edge and join consecutive vertices in each face. This is the **medial graph**. The vertices of the planar graph give a 2-coloring. Choosing the other color gives the planar dual.

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

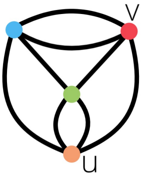
Grid Theorem (Robertson-Seymour)

A class of graphs has unbounded branch-width iff it has all planar graphs as minors.



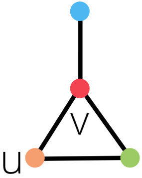
planar graph

$\sim$



tour graph

$\sim$



circle graph

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

Grid Theorem (Robertson-Seymour)

A class of graphs has unbounded branch-width iff it has all planar graphs as minors.

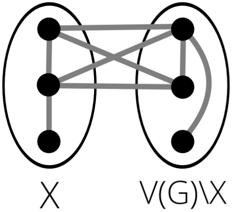
Theorem (Geelen-Kwon-McCarty-Wollan)

A class of graphs has unbounded **rank-width** *iff it has all* **circle graphs** *as* **vertex-minors***.*

Conjectured by Oum.

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

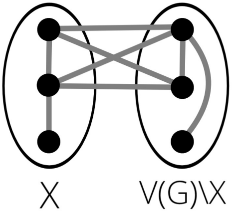
$$
\begin{array}{cc}
 & \begin{array}{cc} \quad X \quad & \quad V(G) \setminus X \end{array} \\
\begin{array}{c} X \\ \\ V(G) \setminus X \end{array} &
\left[\begin{array}{ccc|ccc}
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
$$



(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

$$
\begin{array}{cc}
 & \begin{array}{cc} \quad X \quad & \quad V(G) \setminus X \end{array} \\
\begin{array}{c} X \\ \\ V(G) \setminus X \end{array} &
\left[\begin{array}{ccc|ccc}
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
$$

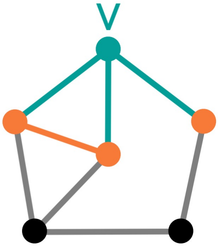


$$\textbf{cut-rank}(X) = \textbf{cut-rank}(V(G) \setminus X)$$

(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

$$
\begin{array}{cc}
 & V(G) \\
V(G) & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}
\end{array}
$$



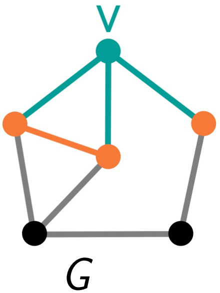
G

Cut-rank(X) is invariant under local complementation.

(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

$$\begin{array}{cc} & V(G) \\ V(G) & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdot & 1 & 0 & 1 & 0 \\ 1 & 1 & \cdot & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdot & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

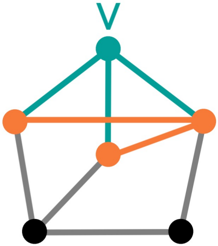


Cut-rank(X) is invariant under local complementation.

(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

$$\begin{array}{cc} & V(G) \\ V(G) & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & \cdot & 0 & 1 & 1 & 0 \\ 1 & 0 & \cdot & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdot & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{array}{l} \\ -\mathrm{I} \\ -\mathrm{I} \\ -\mathrm{I} \\ \\ \\ \end{array} \end{array}$$



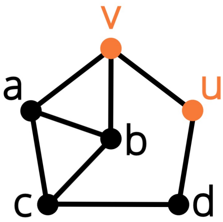
$G * v$

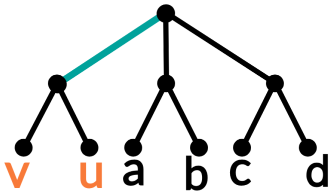
Cut-rank(X) is invariant under local complementation.

(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

Rank-width(G) is the minimum **width** of a subcubic tree T with leafs $V(G)$.





$$\textbf{width}(T) = \max_{e \in E(T)} \text{cut-rank}(X_e)$$

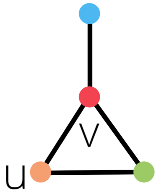
(Oum-Seymour, Bouchet, Oum)

For $X \subseteq V(G)$, **cut-rank**(X) is the rank over the binary field of...

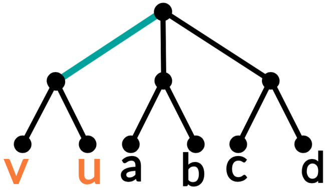
Rank-width(G) is the minimum **width** of a subcubic tree T with leafs $V(G)$.

Theorem (Geelen-Kwon-McCarty-Wollan)

A class of graphs has unbounded **rank-width** *iff it has all* **circle graphs** *as* **vertex-minors**.



circle graph

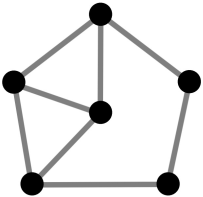


decomposition

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

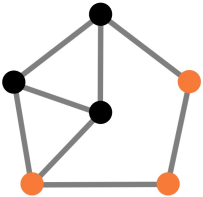
A graph G is a **p-perturbation** of G' if the diagonal of $\mathrm{Adj}(G) + \mathrm{Adj}(G')$ can be filled in to rank $\leq p$.



Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

A graph G is a p-**perturbation** of G' if the diagonal of $\mathrm{Adj}(G) + \mathrm{Adj}(G')$ can be filled in to rank $\leq p$.



Conjecture (Geelen)

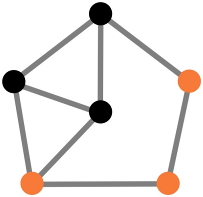
The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

A graph G is a p-**perturbation** of G' if the diagonal of $\mathrm{Adj}(G) + \mathrm{Adj}(G')$ can be filled in to rank $\leq p$.

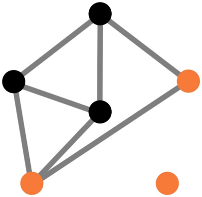
Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

A graph G is a ***p*-perturbation** of G' if the diagonal of $\mathrm{Adj}(G) + \mathrm{Adj}(G')$ can be filled in to rank $\leq p$.



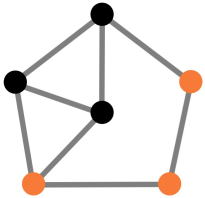
$$\begin{array}{cc} & V(G) \\ V(G) & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



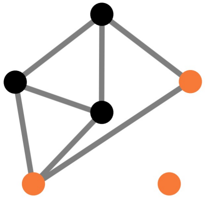
Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

A graph G is a p-**perturbation** of G' if the diagonal of $\mathrm{Adj}(G) + \mathrm{Adj}(G')$ can be filled in to rank $\leq p$.



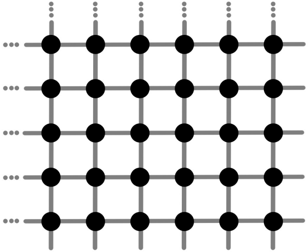
$$\begin{array}{cc} & V(G) \\ V(G) & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$



Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

WMA our favorite circle graph is an induced subgraph.

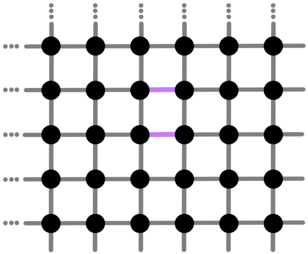


Say one whose **tour graph** has a big grid subgraph.

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

WMA our favorite circle graph is an induced subgraph.

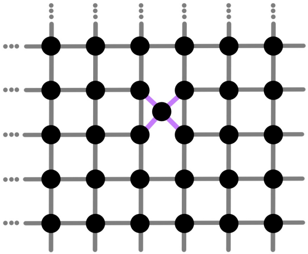


Say one whose **tour graph** has a big grid subgraph.
Add in vertices for as long as possible.

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

WMA our favorite circle graph is an induced subgraph.

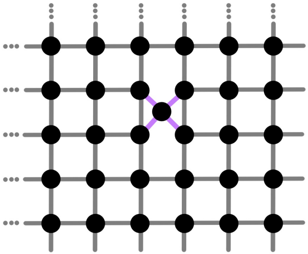


Say one whose **tour graph** has a big grid subgraph.
Add in vertices for as long as possible.

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

WMA our favorite circle graph is an induced subgraph.

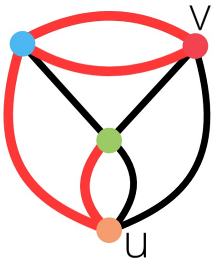


Say one whose **tour graph** has a big grid subgraph.

Add in vertices for as long as possible.

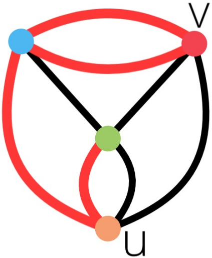
When can we add in x?

When can we add in x?



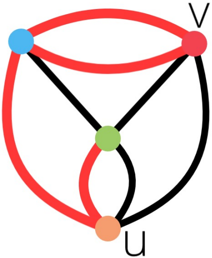
- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.

When can we add in x?



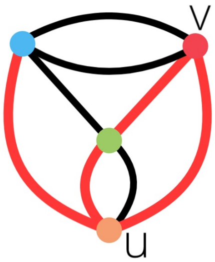
- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.
- We view it as a **signed graph**,

When can we add in x?



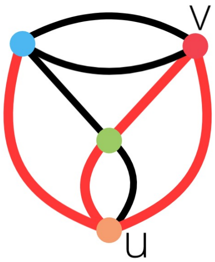
- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.
- We view it as a **signed graph**, so we can **shift** at vertices.

When can we add in x?



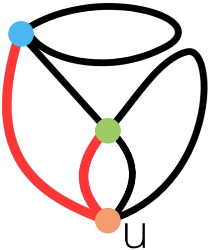
- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.
- We view it as a **signed graph**, so we can **shift** at vertices.

When can we add in x?



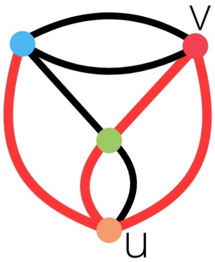
- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.
- We view it as a **signed graph**, so we can **shift** at vertices.
- To **split off**, we "add the parities".

When can we add in x?



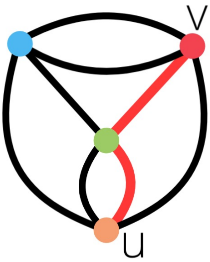
- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.
- We view it as a **signed graph**, so we can **shift** at vertices.
- To **split off**, we "add the parities".

When can we add in x?



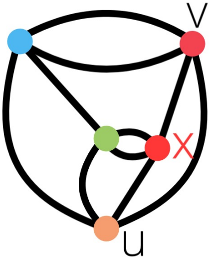
- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.
- We view it as a **signed graph**, so we can **shift** at vertices.
- To **split off**, we "add the parities".
- We can add $x \iff$ we can **shift** so that $|\Sigma_x| \leq 2$ (Bouchet).

When can we add in x?



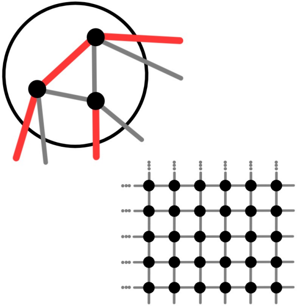
- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.
- We view it as a **signed graph**, so we can **shift** at vertices.
- To **split off**, we "add the parities".
- We can add $x \iff$ we can **shift** so that $|\Sigma_x| \leq 2$ (Bouchet).

When can we add in x?



- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.
- We view it as a **signed graph**, so we can **shift** at vertices.
- To **split off**, we "add the parities".
- We can add $x \iff$ we can **shift** so that $|\Sigma_x| \leq 2$ (Bouchet).

When can we add in x?



- The neighborhood of x can be stored as $\Sigma_x \subseteq E($**tour graph**$)$ of even size.
- We view it as a **signed graph**, so we can **shift** at vertices.
- To **split off**, we "add the parities".
- We can add $x \iff$ we can **shift** so that $|\Sigma_x| \leq 2$ (Bouchet).

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

Conjecture (Oum)

Every **vertex-minor***-closed class has* **finitely many** *minimal forbidden vertex-minors.*

Conjecture (folklore)

Can test if an n-vertex graph has an H-**vertex-minor** *in time* $poly_{\mathrm{H}}(n)$.

Conjecture (Esperet; see Davies)

The chromatic number of any graph with clique number ω *and no* H-**vertex-minor** *is* $\leq poly_{\mathrm{H}}(\omega)$

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

Conjecture (Oum)

Every **vertex-minor***-closed class has* **finitely many** *minimal forbidden vertex-minors.*

Conjecture (folklore)

*Can test if an n-vertex graph has an H-***vertex-minor** *in time $poly_{\mathrm{H}}(n)$.*

Conjecture (Esperet; see Davies)

*The chromatic number of any graph with clique number ω and no H-***vertex-minor** *is $\leq poly_{\mathrm{H}}(\omega)$*

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

Conjecture (Oum)

Every **vertex-minor***-closed class has* **finitely many** *minimal forbidden vertex-minors.*

Conjecture (folklore)

Can test if an n-vertex graph has an H**-vertex-minor** *in time* $poly_{\mathrm{H}}(n)$.

Conjecture (Esperet; see Davies)

The chromatic number of any graph with clique number ω *and no* H**-vertex-minor** *is* $\leq poly_{\mathrm{H}}(\omega)$

Conjecture (Geelen)

The graphs in any proper **vertex-minor***-closed class "decompose" into parts that are "almost"* **circle graphs***.*

Conjecture (Oum)

Every **vertex-minor***-closed class has* **finitely many** *minimal forbidden vertex-minors.*

Conjecture (folklore)

*Can test if an n-vertex graph has an H-***vertex-minor** *in time $poly_{\mathrm{H}}(n)$.*

Conjecture (Esperet; see Davies)

*The chromatic number of any graph with clique number ω and no H-***vertex-minor** *is $\leq poly_{\mathrm{H}}(\omega)$*

Thank you!