Local structure for vertex-minors

Rose McCarty

Department of Combinatorics and Optimization

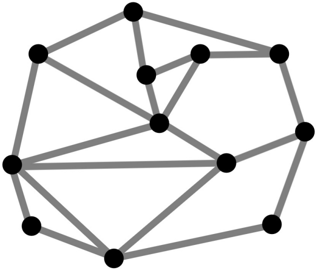
University of Waterloo

September 9th, 2021

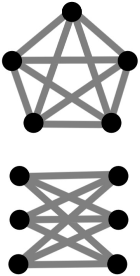
Graph Minor Theorem (Robertson & Seymour 2004)

Every infinite set of graphs contains one that is isomorphic to a minor of another.

Kuratowski's Theorem



planar graphs

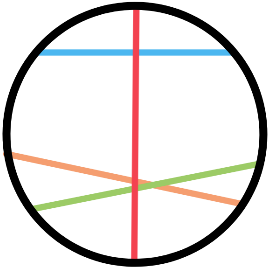


forbidden minors

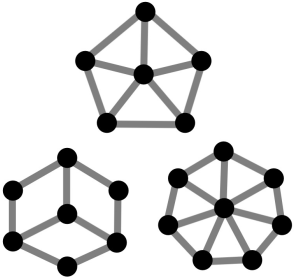
Well-Quasi-Ordering Conjecture (Oum 2017)

Every infinite set of graphs contains one that is isomorphic to a **vertex-minor** *of another.*

Bouchet's Theorem



circle graphs

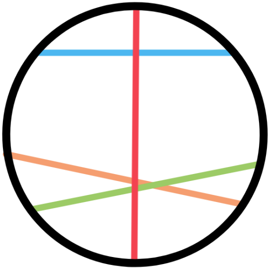


forbidden vertex-minors

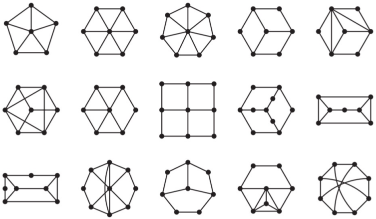
Well-Quasi-Ordering Conjecture (Oum 2017)

Every infinite set of graphs contains one that is isomorphic to a **pivot-minor** *of another.*

Geelen and Oum's Theorem



circle graphs



forbidden pivot-minors

Common generalization! (Bouchet 1988; de Fraysseix 1981)

Structure Theorem (Robertson & Seymour 2003)

For any proper minor-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that "almost embed" in a surface of bounded genus.

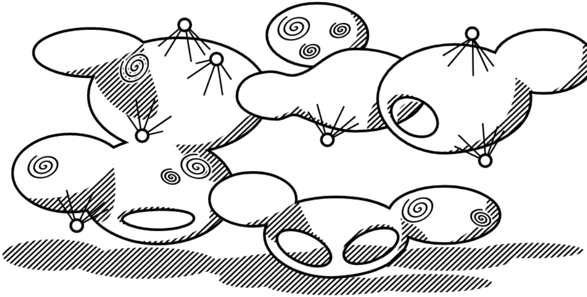


figure by Felix Reidl

Structure Theorem (Robertson & Seymour 2003)

For any proper minor-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that "almost embed" in a surface of bounded genus.

Used to prove:

- Well-Quasi-Ordering (Robertson & Seymour 2004)

Structure Theorem (Robertson & Seymour 2003)

For any proper minor-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that "almost embed" in a surface of bounded genus.

Used to prove:

- Well-Quasi-Ordering (Robertson & Seymour 2004)
- Can determine if $G \in \mathcal{F}$ in polynomial time. (Robertson & Seymour 1995)

Structure Theorem (Robertson & Seymour 2003)

For any proper minor-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that "almost embed" in a surface of bounded genus.

Used to prove:

- Well-Quasi-Ordering (Robertson & Seymour 2004)
- Can determine if $G \in \mathcal{F}$ in polynomial time. (Robertson & Seymour 1995)
- Can 2-approximate the chromatic number of $G \in \mathcal{F}$. (Demaine, Hajiaghayi, & Kawarabayashi 2005)

Structure Theorem (Robertson & Seymour 2003)

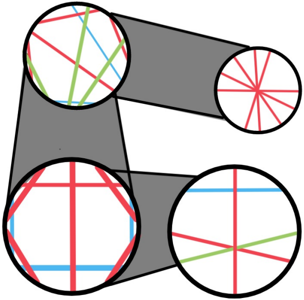
For any proper minor-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that "almost embed" in a surface of bounded genus.

Used to prove:

- Well-Quasi-Ordering (Robertson & Seymour 2004)
- Can determine if $G \in \mathcal{F}$ in polynomial time. (Robertson & Seymour 1995)
- Can 2-approximate the chromatic number of $G \in \mathcal{F}$. (Demaine, Hajiaghayi, & Kawarabayashi 2005)
- First-order model-checking is FPT on $G \in \mathcal{F}$. (Flum & Grohe 2007)

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that are "almost"* **circle graphs***.*



The thesis is part of an ongoing project with
Jim Geelen & Paul Wollan to prove the conjecture.

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class* $\mathcal{F}$*, each* $G \in \mathcal{F}$ *"decomposes" into parts that are "almost"* **circle graphs***.*

Conjectures:

- Well-quasi-ordering (Oum)

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that are "almost"* **circle graphs***.*

Conjectures:

- Well-quasi-ordering (Oum)
- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that are "almost"* **circle graphs***.*

Conjectures:

- Well-quasi-ordering (Oum)
- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number ω of $G \in \mathcal{F}$ in polynomial time (Geelen),

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class* $\mathcal{F}$*, each* $G \in \mathcal{F}$ *"decomposes" into parts that are "almost"* **circle graphs***.*

Conjectures:

- Well-quasi-ordering (Oum)
- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number ω of $G \in \mathcal{F}$ in polynomial time (Geelen), and the chromatic number of G is $\leq \mathrm{poly}(\omega)$. (Davies; Esperet; Kim, Kwon, Oum, & Sivaraman)

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that are "almost"* **circle graphs***.*

Conjectures:

- Well-quasi-ordering (Oum)
- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number ω of $G \in \mathcal{F}$ in polynomial time (Geelen), and the chromatic number of G is $\leq \mathrm{poly}(\omega)$. (Davies; Esperet; Kim, Kwon, Oum, & Sivaraman)
- Can $n^{1-\epsilon}$-approximate the chromatic number of $G \in \mathcal{F}$.

Structural Conjecture (Geelen)

*For any proper **vertex-minor**-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that are "almost" **circle graphs**.*

Conjectures:

- Well-quasi-ordering (Oum)
- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number ω of $G \in \mathcal{F}$ in polynomial time (Geelen), and the chromatic number of G is $\leq \mathrm{poly}(\omega)$. (Davies; Esperet; Kim, Kwon, Oum, & Sivaraman)
- Can $n^{1-\epsilon}$-approximate the chromatic number of $G \in \mathcal{F}$.
- For any ω, first-order model-checking is FPT on $G \in \mathcal{F}$ with clique number $\leq \omega$.

Structural Conjecture (Geelen)

For any proper **vertex-minor***-closed class $\mathcal{F}$, each $G \in \mathcal{F}$ "decomposes" into parts that are "almost"* **circle graphs***.*

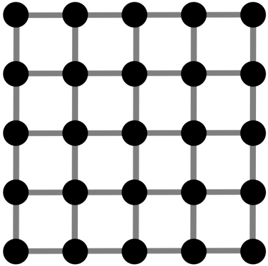
Conjectures:

- Well-quasi-ordering (Oum)
- Can determine if $G \in \mathcal{F}$ in polynomial time. (Oum)
- Can determine the clique number ω of $G \in \mathcal{F}$ in polynomial time (Geelen), and the chromatic number of G is $\leq \mathrm{poly}(\omega)$. (Davies; Esperet; Kim, Kwon, Oum, & Sivaraman)
- Can $n^{1-\epsilon}$-approximate the chromatic number of $G \in \mathcal{F}$.
- For any ω, first-order model-checking is FPT on $G \in \mathcal{F}$ with clique number $\leq \omega$.
- $\mathrm{MBQC}_{\mathcal{F}}$ can be efficiently simulated classically. (Geelen)

Grid Theorem (Robertson & Seymour 1986)

For any planar graph H, every graph with tree-width $\geq f(H)$ has a minor isomorphic to H.

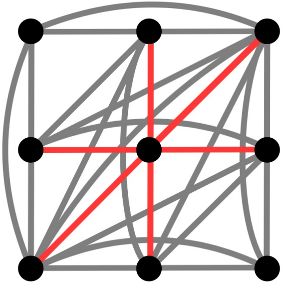
If you cannot "decompose away the whole graph", then there is a big grid as a minor.



Theorem (Geelen, Kwon, McCarty, & Wollan 2020)

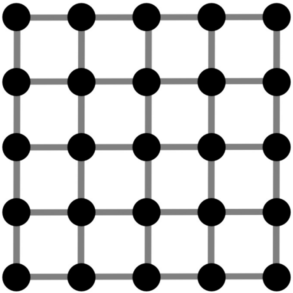
For any **circle graph** *H, every graph with* **rank-width** *$\geq f(H)$ has a* **vertex-minor** *isomorphic to H.*

If you cannot "rank-decompose away the whole graph", then there is a big **comparability grid** as a vertex-minor.



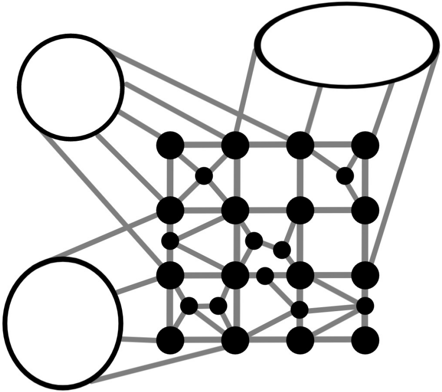
Flat Wall Theorem (Robertson & Seymour 1995)

For any proper minor-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a large grid minor, there is a planar subgraph containing a lot of the grid so that the rest of G "almost attaches" onto just the outer face.



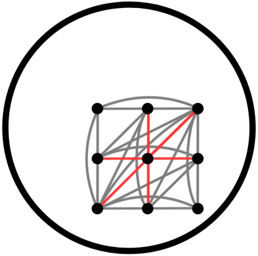
Flat Wall Theorem (Robertson & Seymour 1995)

For any proper minor-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a large grid minor, there is a planar subgraph containing a lot of the grid so that the rest of G "almost attaches" onto just the outer face.



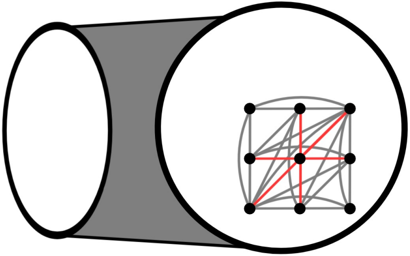
Local Structure Theorem (Geelen, McCarty, & Wollan)

For any proper **vertex-minor***-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a prime* **circle graph** *containing a* **comparability grid**, *the rest of G "almost attaches" in a way that is "mostly compatable".*

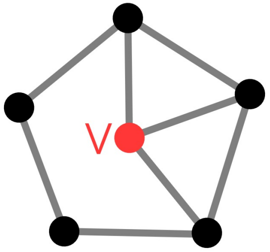


Local Structure Theorem (Geelen, McCarty, & Wollan)

For any proper **vertex-minor***-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a prime* **circle graph** *containing a* **comparability grid***, the rest of G "almost attaches" in a way that is "mostly compatable".*

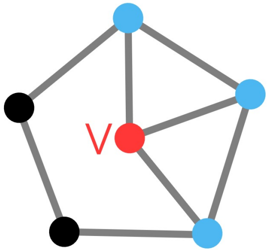


Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement.



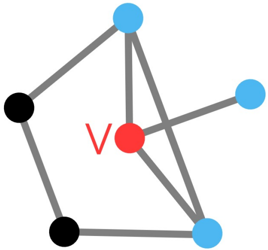
G

Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement.



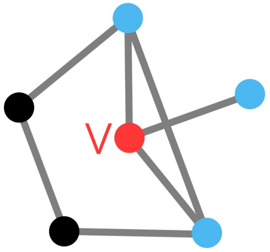
G

Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement.



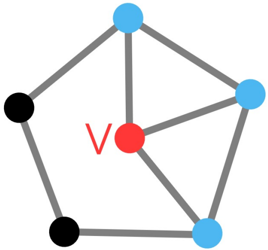
$G * v$

Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement. This yields an equivalence relation called **local equivalence**.



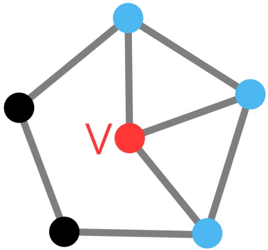
$G * v$

Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement. This yields an equivalence relation called **local equivalence**.



$G * v * v$

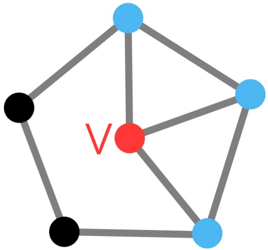
Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement. This yields an equivalence relation called **local equivalence**.



G

Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement. This yields an equivalence relation called **local equivalence**.

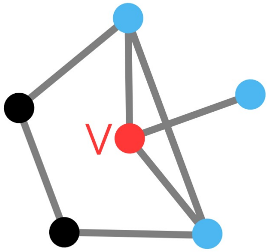
The **vertex-minors** of G are the induced subgraphs of graphs in its local equivalence class.



G

Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement. This yields an equivalence relation called **local equivalence**.

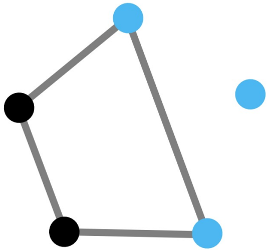
The **vertex-minors** of G are the induced subgraphs of graphs in its local equivalence class.



$G * v$

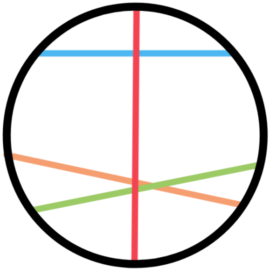
Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement. This yields an equivalence relation called **local equivalence**.

The **vertex-minors** of G are the induced subgraphs of graphs in its local equivalence class.

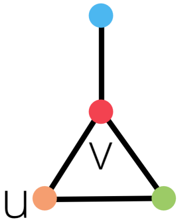


$G * v - v$

A **circle graph** is the intersection graph of chords on a circle.

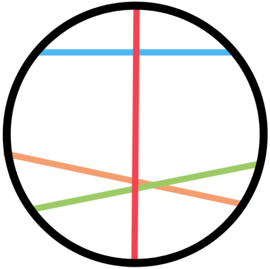


chord diagram

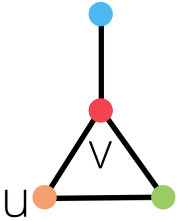


circle graph G

A **circle graph** is the intersection graph of chords on a circle. They are closed under local complementation.

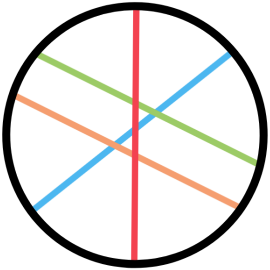


chord diagram

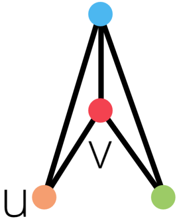


circle graph G

A **circle graph** is the intersection graph of chords on a circle. They are closed under local complementation.

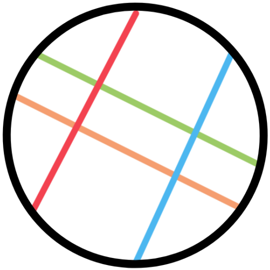


chord diagram

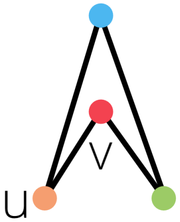


circle graph $G * v$

A **circle graph** is the intersection graph of chords on a circle.
They are closed under local complementation.

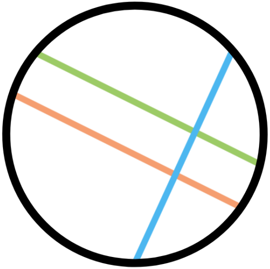


chord diagram

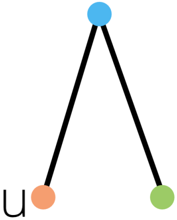


circle graph $G * v * u$

A **circle graph** is the intersection graph of chords on a circle. They are closed under local complementation.

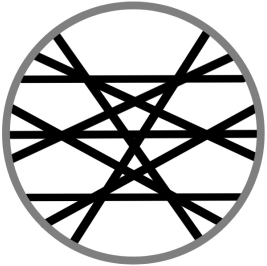


chord diagram

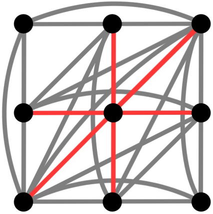


circle graph $G * v * u - v$

A **circle graph** is the intersection graph of chords on a circle. They are closed under local complementation. Every circle graph is a vertex-minor of a comparability grid.

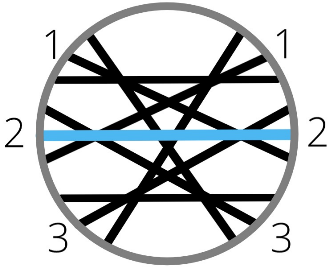


chord diagram

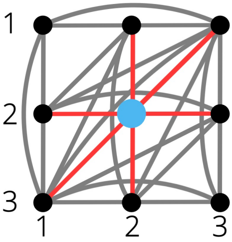


comparability grid

A **circle graph** is the intersection graph of chords on a circle. They are closed under local complementation. Every circle graph is a vertex-minor of a comparability grid.



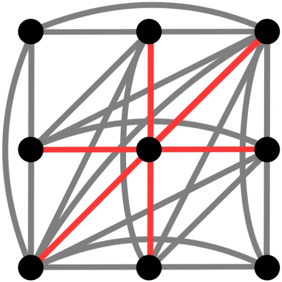
chord diagram



comparability grid

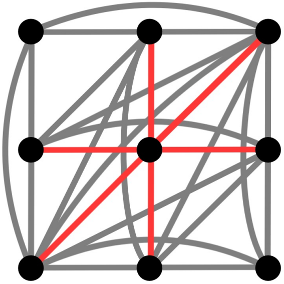
To prove the structural conjecture...

- WMA a big **comparability grid** is a **vertex-minor**.



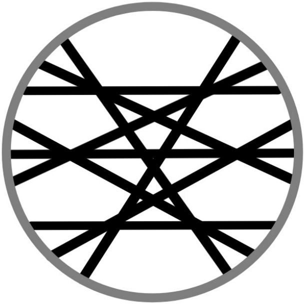
To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.



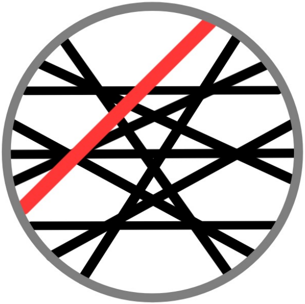
To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.



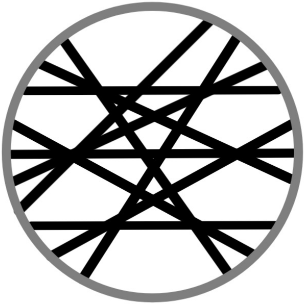
To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.



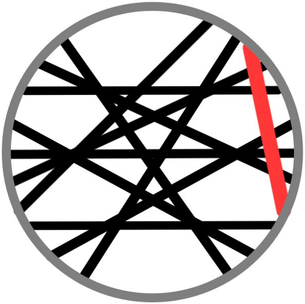
To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.



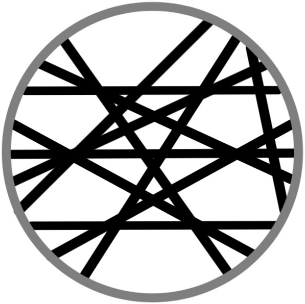
To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.



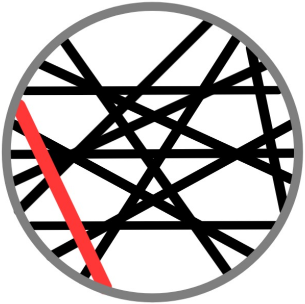
To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.



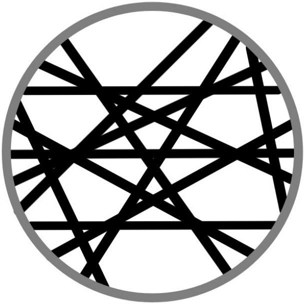
To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.



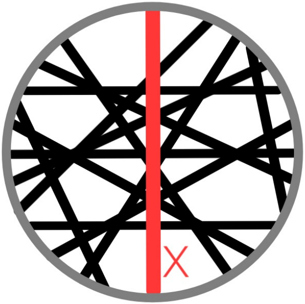
To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?



To prove the structural conjecture...

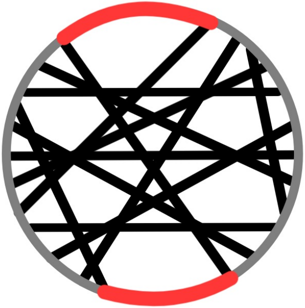
- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?



If x can be added as a chord,

To prove the structural conjecture...

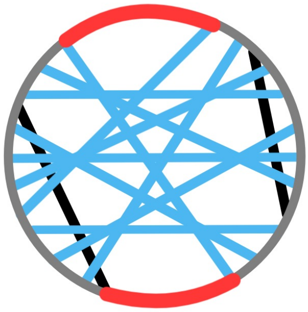
- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?



If x can be added as a chord,
then its neighbourhood can be encoded by two arcs.

To prove the structural conjecture...

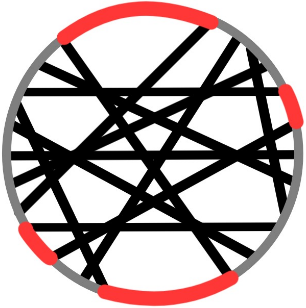
- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?



If x can be added as a chord,
then its neighbourhood can be encoded by two arcs.

To prove the structural conjecture...

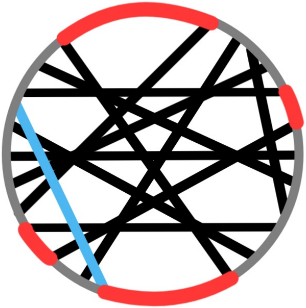
- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?



Any neighbourhood can be encoded by an even number of arcs.

To prove the structural conjecture...

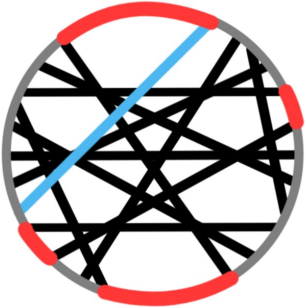
- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?



Any neighbourhood can be encoded by an even number of arcs.

To prove the structural conjecture...

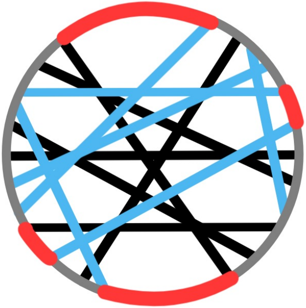
- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?



Any neighbourhood can be encoded by an even number of arcs.

To prove the structural conjecture...

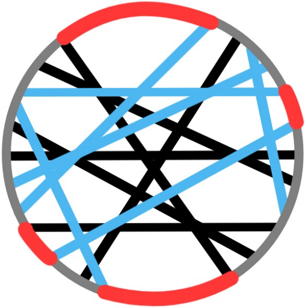
- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?



Any neighbourhood can be encoded by an even number of arcs.

To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?

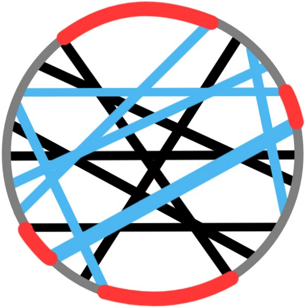


Any neighbourhood can be encoded by an even number of arcs.

These arc-sets are called **signatures** and are equiv up to **shifting**.

To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?

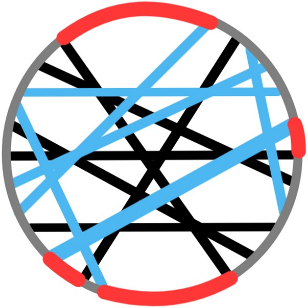


Any neighbourhood can be encoded by an even number of arcs.

These arc-sets are called **signatures** and are equiv up to **shifting**.

To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?

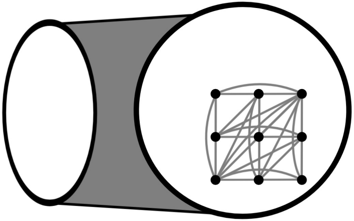


Any neighbourhood can be encoded by an even number of arcs.

These arc-sets are called **signatures** and are equiv up to **shifting**.

To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?

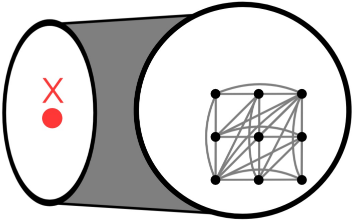


Any neighbourhood can be encoded by an even number of arcs.

These arc-sets are called **signatures** and are equiv up to **shifting**.

To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?

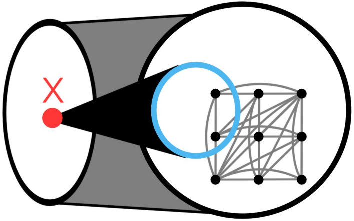


Any neighbourhood can be encoded by an even number of arcs.

These arc-sets are called **signatures** and are equiv up to **shifting**.

To prove the structural conjecture...

- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?

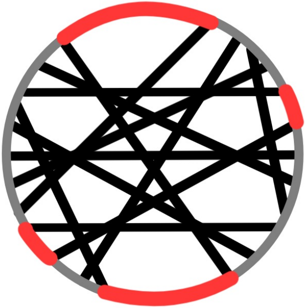


Any neighbourhood can be encoded by an even number of arcs.

These arc-sets are called **signatures** and are equiv up to **shifting**.

To prove the structural conjecture...

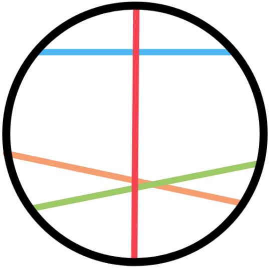
- WMA a big **comparability grid** is an induced subgraph.
- Find a prime **circle graph** containing the comparability grid.
- How can we tell if x can be added to the circle graph?



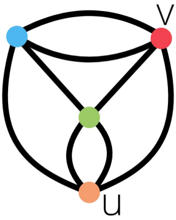
Any neighbourhood can be encoded by an even number of arcs.

These arc-sets are called **signatures** and are equiv up to **shifting**.

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.

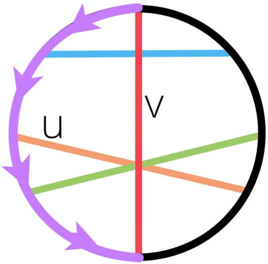


chord diagram

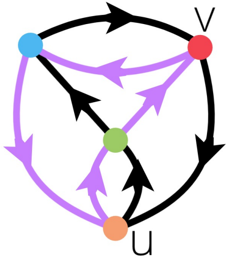


tour graph

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.

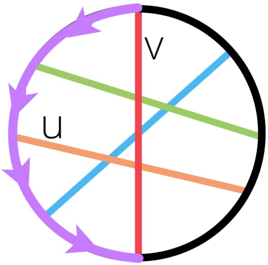


chord diagram

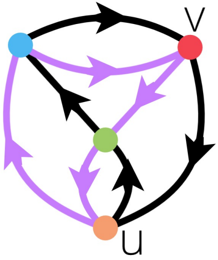


tour graph

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.

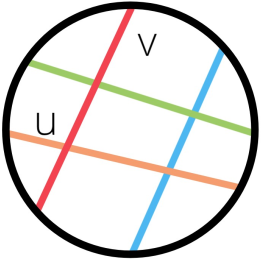


chord diagram

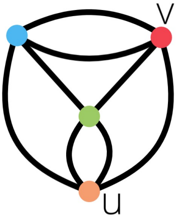


tour graph

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.

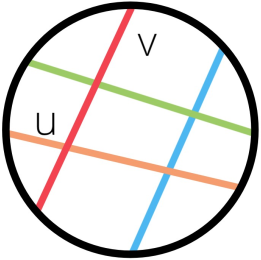


chord diagram

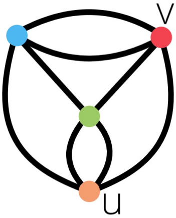


tour graph

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.

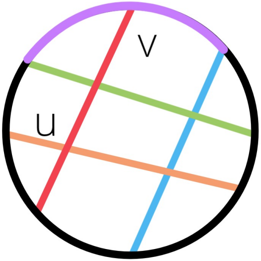


chord diagram

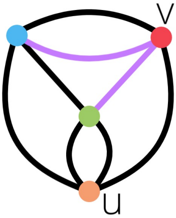


tour graph

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.

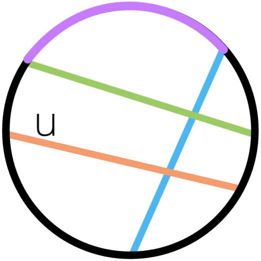


chord diagram

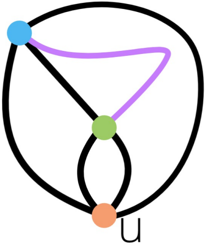


tour graph

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.

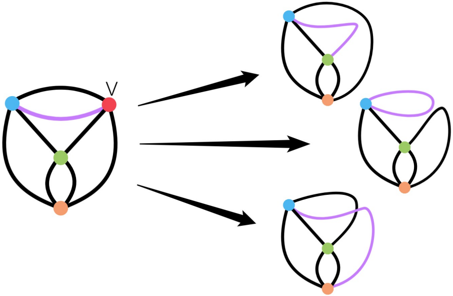


chord diagram



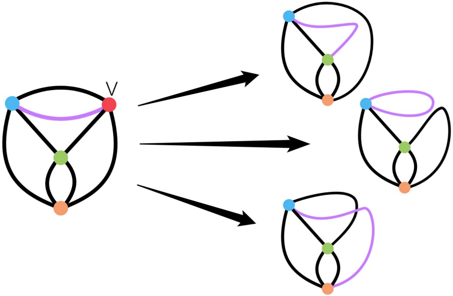
tour graph

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.



There are 3 ways to pair the incident edges; this defines **complete immersion minors**.

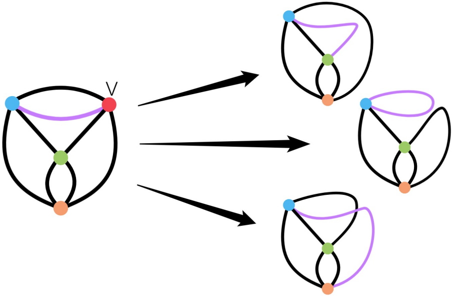
View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.



Theorem (Kotzig 1968; Bouchet 1987)

If H and G are prime circle graphs, then their tour graphs are unique, and H is a **vertex-minor** *of G iff its tour graph is a* **complete immersion minor** *of the tour graph of G.*

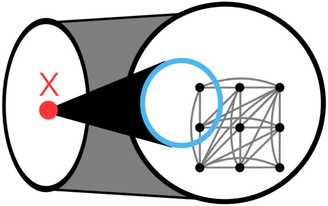
View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.



Theorem (Kotzig 1968; Bouchet 1987)

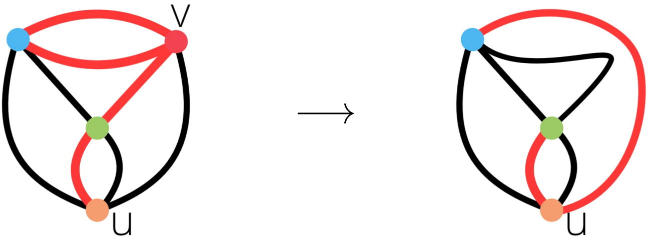
If H and G are prime circle graphs, then their tour graphs are unique, and H is a **vertex-minor** *of G iff its tour graph is a* **complete immersion minor** *of the tour graph of G.*

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.



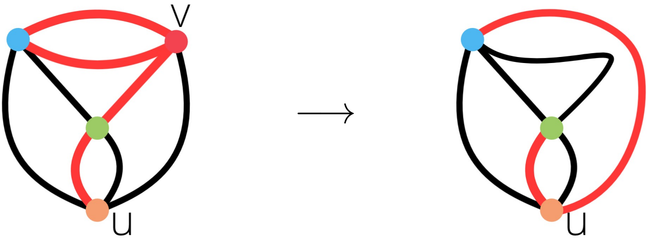
Something similar holds with **signatures** when we only allow local complementations at vertices in the circle graph.

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.



Something similar holds with **signatures** when we only allow local complementations at vertices in the circle graph.

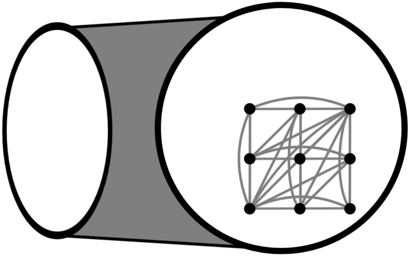
View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation, and vertex-deletion works nicely.



Something similar holds with **signatures** when we only allow local complementations at vertices in the circle graph. (Primeness just makes the **tour graph** unique.)

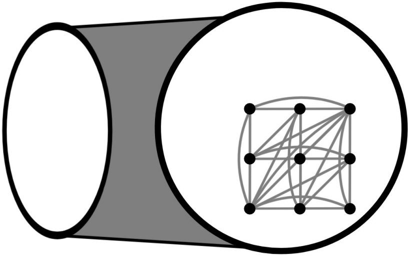
Local Structure Theorem (Geelen, McCarty, & Wollan)

For any proper **vertex-minor***-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a prime* **circle graph** *containing a* **comparability grid***, the rest of G "almost attaches" in a way that is "mostly compatable".*



Local Structure Theorem (Geelen, McCarty, & Wollan)

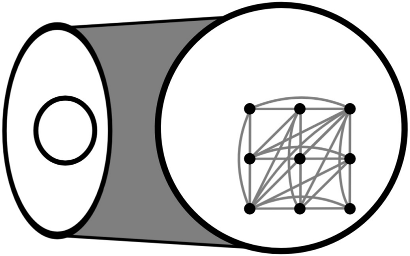
For any proper **vertex-minor***-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a prime* **circle graph** *containing a* **comparability grid***, the rest of G "***almost attaches***" in a way that is "mostly compatable".*



After a small **perturbation**,

Local Structure Theorem (Geelen, McCarty, & Wollan)

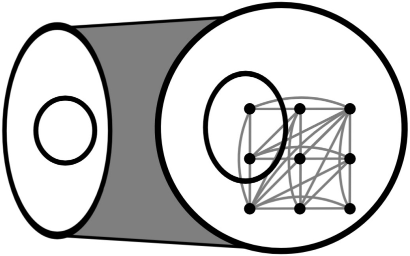
For any proper **vertex-minor***-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a prime* **circle graph** *containing a* **comparability grid***, the rest of G "***almost attaches***" in a way that is "mostly compatable".*



After a small **perturbation**,

Local Structure Theorem (Geelen, McCarty, & Wollan)

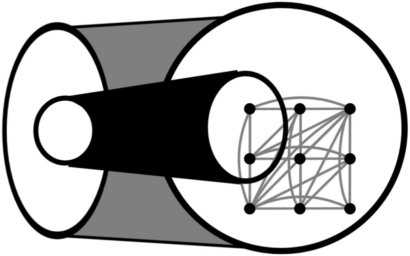
For any proper **vertex-minor***-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a prime* **circle graph** *containing a* **comparability grid***, the rest of G "***almost attaches***" in a way that is "mostly compatable".*



After a small **perturbation**,

Local Structure Theorem (Geelen, McCarty, & Wollan)

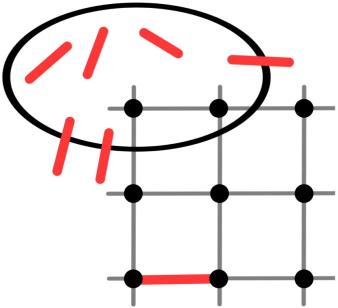
For any proper **vertex-minor***-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a prime* **circle graph** *containing a* **comparability grid***, the rest of G "***almost attaches***" in a way that is "mostly compatable".*



After a small **perturbation**,

Local Structure Theorem (Geelen, McCarty, & Wollan)

For any proper **vertex-minor***-closed class $\mathcal{F}$ and any $G \in \mathcal{F}$ with a prime* **circle graph** *containing a* **comparability grid***, the rest of G "almost attaches" in a way that is "***mostly compatable***".*



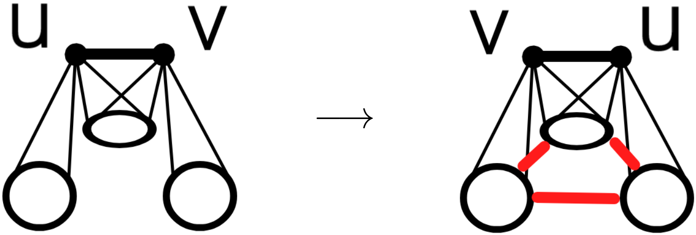
After a small **perturbation**, each **signature** of the tour graph is on the "wrong side" of a small cut wrt the comparability grid.

Future work:

- Structure
- Well-quasi-ordering, membership testing, colouring, . . .
- **Pivot-minors**?

Future work:

- Structure
- Well-quasi-ordering, membership testing, colouring, ...
- **Pivot-minors**?



G

$$G * u * v * u = G * v * u * v$$