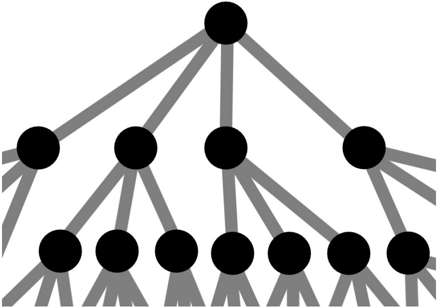
Unavoidable dense induced subgraphs

Rose McCarty

September 2020

Theorem (Erdös 1959)

For any d, k, there exists a graph with avgdeg $\geq d$ and girth $\geq k$.



locally such a graph is a tree

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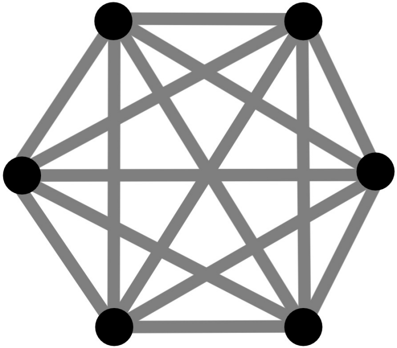

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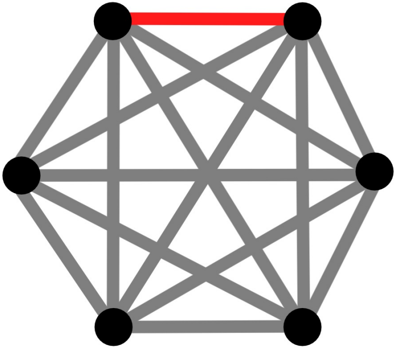

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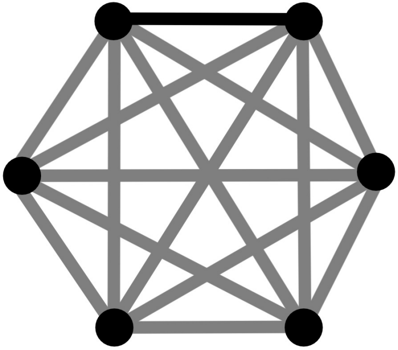

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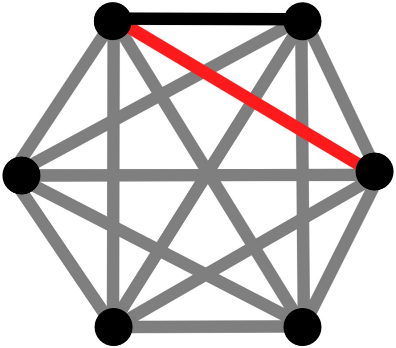

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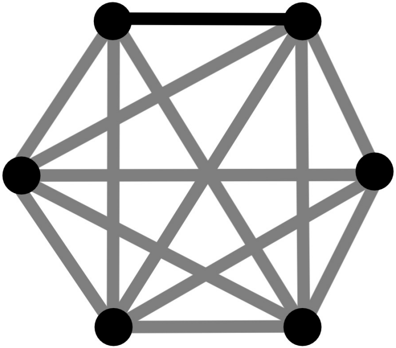

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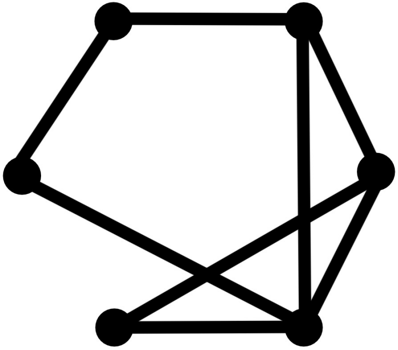

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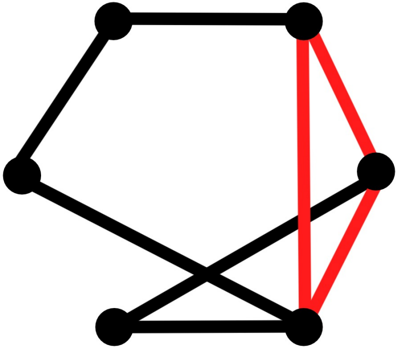

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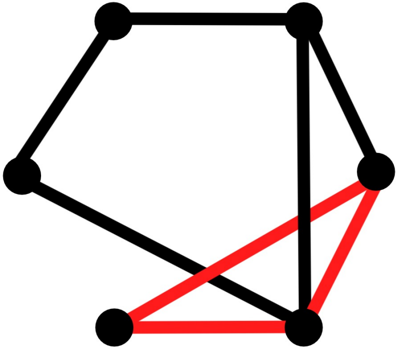

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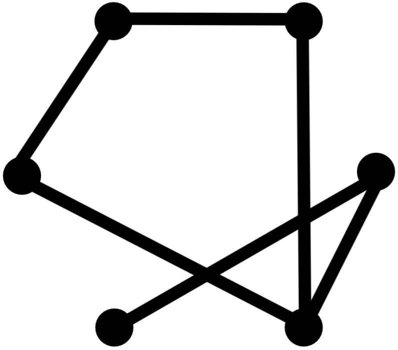
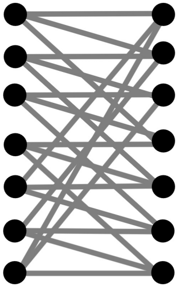


Conjecture (Thomassen, 1983)

For any d, k, **every** *graph of avgdeg $\geq f(d,k)$ has a subgraph with avgdeg $\geq d$ and girth $\geq k$.*



avgdeg $\geq d$, girth $\geq k$

This notation is used throughout the talk to mean that "For any d, k, there exists $f(d,k)$ such that..."

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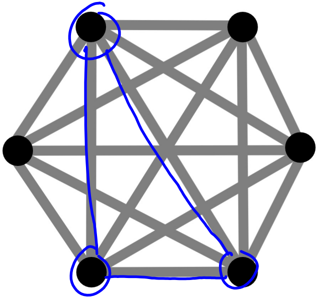
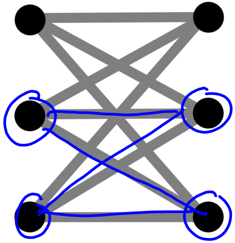
- The previous approach works for regular graphs,
- but there are graphs with avgdeg $\geq D$ and no d-regular subgraph, for any $D \geq d \geq 3$. (Pyber, Rödl, Szemerédi, 1995).

What if we want an **induced** subgraph of large average degree and girth?



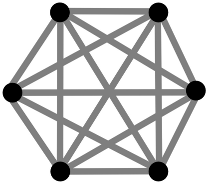
clique K_6



biclique $K_{3,3}$

Conjecture

For any d, k, **every** *graph of avgdeg $\geq f(d,k)$ has either K_d, or $K_{d,d}$, or a graph with avgdeg $\geq d$ and girth $\geq k$ as an* **induced** *subgraph.*



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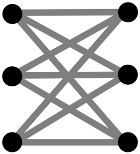

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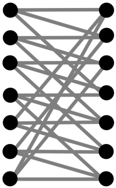

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- Implies Thomassen's Conjecture.

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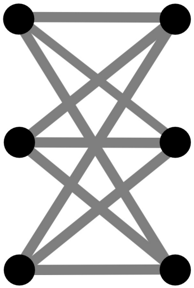

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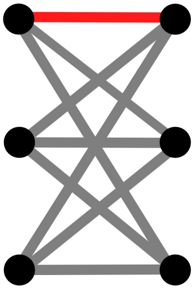

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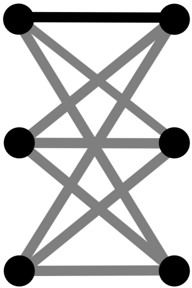

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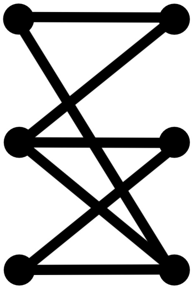

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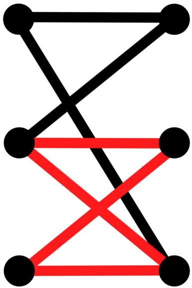

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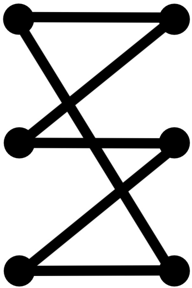

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- Implies Thomassen's Conjecture.
- Why should this conjecture be true?

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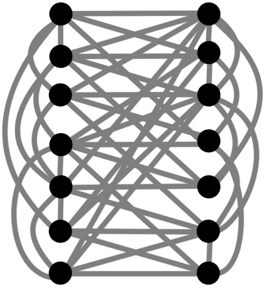
- Implies Thomassen's Conjecture.
- Why should this conjecture be true?
- What makes Thomassen's Conjecture hard?

Reduction to bipartite graphs

Proposition

For any d, **every** *graph of avgdeg $\geq 2d$ has a* **bipartite** *subgraph with avgdeg $\geq d$.*



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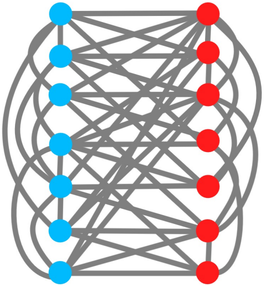

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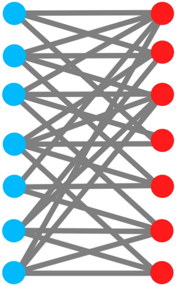
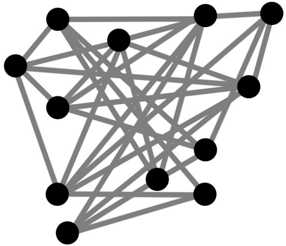


Reduction to bipartite graphs

Theorem (Kwan, Letzter, Sudakov, and Tran, 2020)

For any d, **every** *graph of avgdeg $\geq f(d)$ has either K_d or a* **bipartite** *graph with avgdeg $\geq d$ as an* **induced** *subgraph.*



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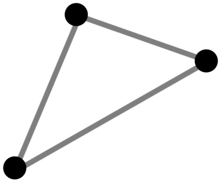

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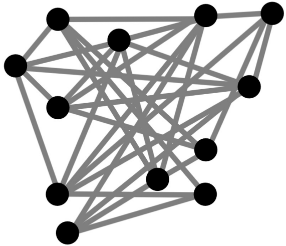

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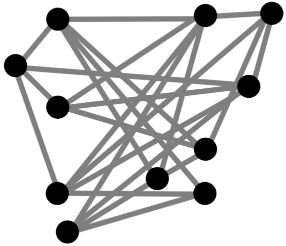

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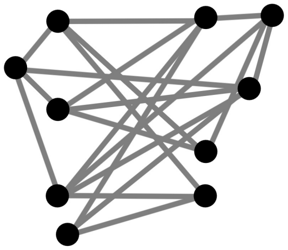

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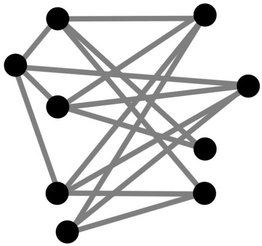

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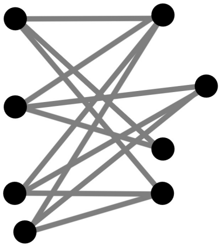
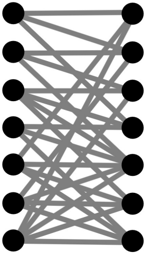


Removing 4-cycles

Theorem (Kühn and Osthus, 2004)

For any d, **every bipartite** *graph of avgdeg $\geq f(d)$ has a subgraph with avgdeg $\geq d$ and* **no** 4-**cycles**.



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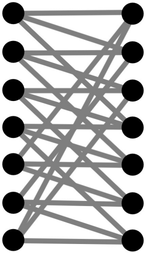

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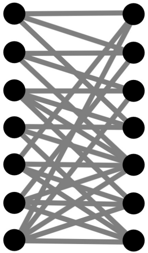
For any d, **every bipartite** *graph of avgdeg $\geq f(d)$ has either $K_{d,d}$ or a graph with avgdeg $\geq d$ and* **no 4-cycles** *as an* **induced** *subgraph.*



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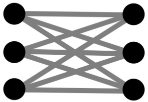

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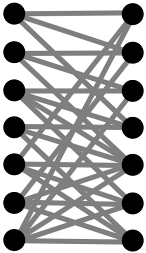

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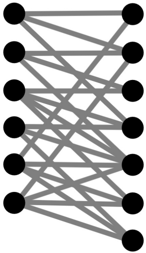

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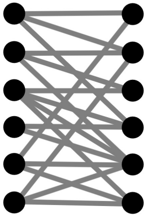

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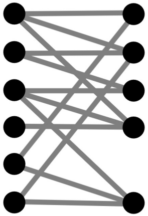

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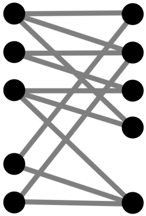

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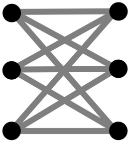
Based on Dellamonica, Koubek, Martin, Rödl, '11

Regular bipartite graphs

Proposition

For any d, k, and $D \geq f(d,k)$, **every** D-**regular bipartite** *graph has either $K_{d,d}$ or a graph with avgdeg $\geq d$ and girth $\geq k$ as an* **induced** *subgraph.*



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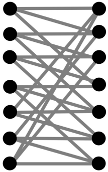

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- Include **vertices** iid and apply Kövári–Sós–Turán to bound the number of short cycles.

Proposition (again)

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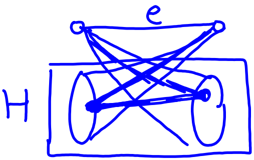
Proof. $(k = 6)$ d, k fixed and $n := |V(G)|/2$

- **Lemma:** Graph has $\mathcal{O}(nD^{3-1/d})$ cycles of length 4, if no $K_{d,d}$.

\# edges $= nD$

\# 4-cycles containing e = \# edges of H

$= O(D^{2-1/d})$



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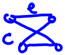
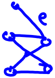
Proof. $(k = 6)$

- **Lemma:** Graph has $\mathcal{O}(nD^{3-1/d})$ cycles of length 4.
- Include each vertex with probability $p = 1/D^{1-\varepsilon}$
  then delete one vertex from each cycle of length 4.

$$\mathbb{E}[\#\text{vertices}] \leq 2pn$$

$$\mathbb{E}[\#\text{edges originally}] = p^2 D n = D^{\varepsilon} p n$$

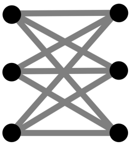
$\mathbb{E}[\#\text{deleted edges}] \lesssim \mathbb{E}[\#(e,C):$  or  $]$

$$\lesssim p^5 n D^{4-1/d} = \frac{pn(pD)^4}{D^{1/d}} = pnD^{4\varepsilon - 1/d}$$

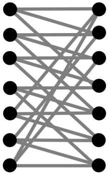
∎

Theorem (again)

For any d, **every bipartite** *graph of avgdeg $\geq f(d)$ has either $K_{d,d}$ or a graph with avgdeg $\geq d$ and* **no** 4-**cycles** *as an* **induced** *subgraph.*



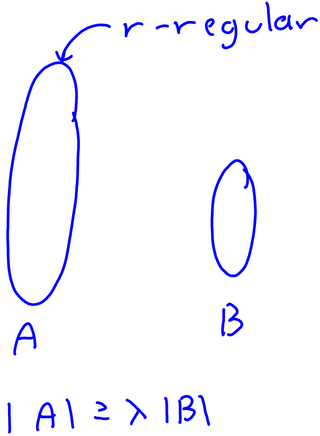
biclique



avgdeg $\geq d$ and girth $\geq k$

Lemma

For any $r, \lambda \geq 1$, **every bipartite** *graph of avgdeg* $\geq f(r, \lambda)$ *has an* **induced** (r, λ)*-subgraph.*



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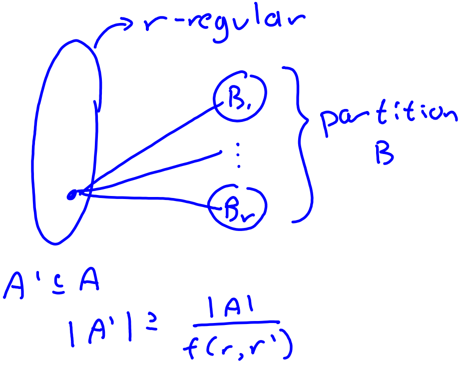
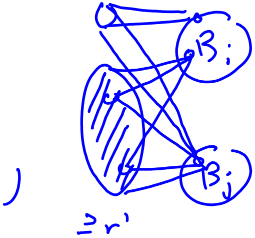
- Apply (Füredi, 1983).





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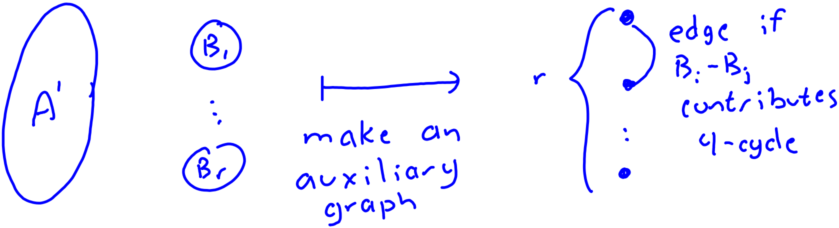


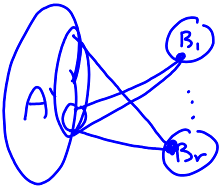
1) large stable set → remove 4-cycles
2) high degree vertex → next page

Lemma

For any $r, \lambda \geq 1$, **every bipartite** *graph of avgdeg $\geq f(r, \lambda)$ has an* **induced** *(r, λ)-subgraph.*

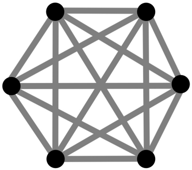
- Apply (Füredi, 1983).

Suppose B_r "adjacent" to all other B_i.

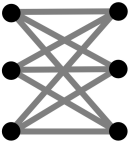


Take any vertex in B_r; every other vertex in B/B_r has at least r' neighbors in common. $\rightarrow$ find $K_{d,d}$ eventually.

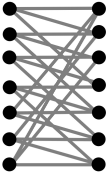
Triangles and 4-cycles are special.



clique

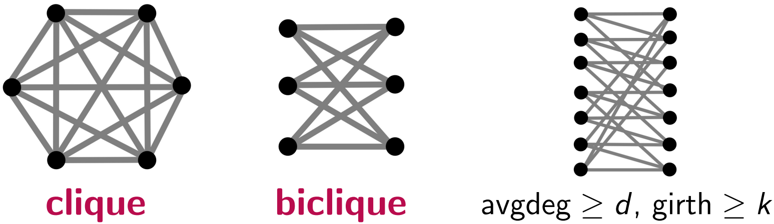


biclique

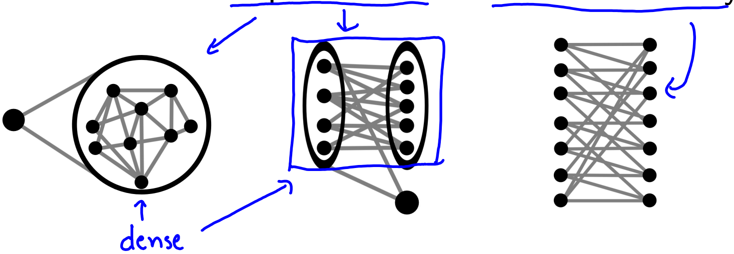


avgdeg $\geq d$, girth $\geq k$

Triangles and 4-cycles are special.



We either find a helpful vertex or win immediately.

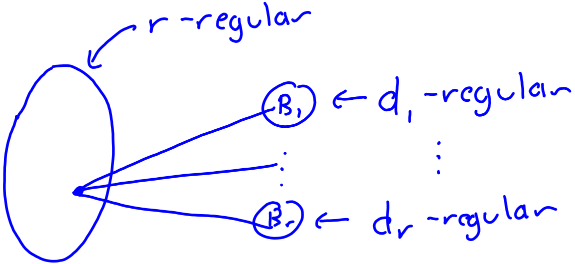


Regular graphs are the crux of the matter...

Theorem (DKMR, '11; Pyber, Rödl, Szemerédi, '95)

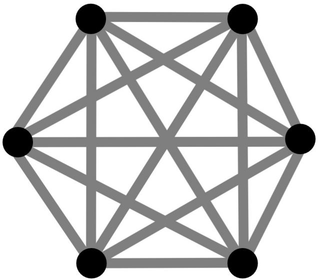
For any D, Δ, there is a graph of avgdeg $\geq D$ with no subgraph of avgdeg ≥ 8 and maximum degree at most Δ.

Do "neighbourhood regular" graphs satisfy the conjecture?

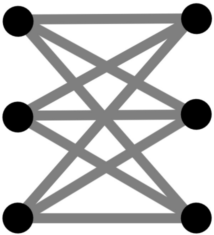


$$r \ll d_1 \ll \dots \ll d_r$$

The Chromatic Number $\chi(G)$



clique $\chi(K_6) = 6$



biclique $\chi(K_{3,3}) = 2$

(so bicliques are not helpful)

The Chromatic Number $\chi(G)$

Theorem (Rödl, 1977)

For any d, **every** *graph with $\chi \geq f(d)$ has a* **triangle-free** *subgraph with $\chi \geq d$.*

Question (Esperet)

Is it true that for any d, **every** *graph with $\chi \geq f(d)$ has either K_d or a* **triangle-free** *graph with $\chi \geq d$ as an* **induced** *subgraph?*

Thank you!