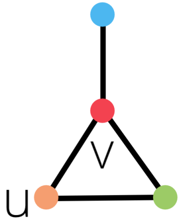
Vertex-minors and flooding immersions

Rose McCarty
Joint work with Jim Geelen and Paul Wollan (ongoing!)

IBS Virtual Discrete Math Colloquium
January 2021

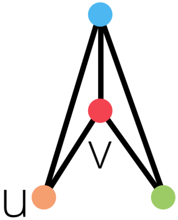


G

Locally complementing at v replaces the induced subgraph on the neighbourhood of v by its complement.

Two graphs are **locally equivalent** if one can be obtained from the other by local complementations.

A graph H is a **vertex-minor** of G if H can be obtained from a graph that is locally equivalent to G by deleting vertices.



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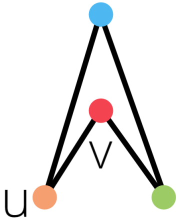

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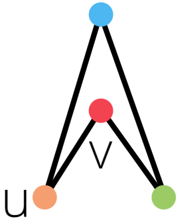

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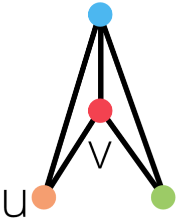

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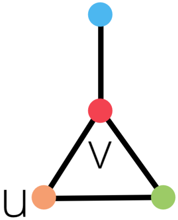

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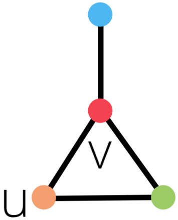

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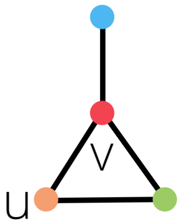

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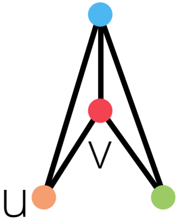

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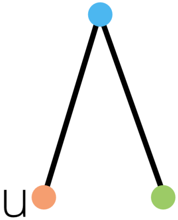

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- Cut-rank: "connectivity for dense graphs" [Bouchet; Oum 05]
- Graph states: "resources in quantum computing" [Raussendorf-Briegel 01, Van den Nest-Dehaene-De Moor 04]
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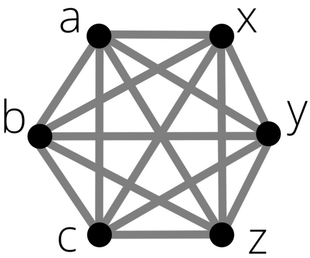

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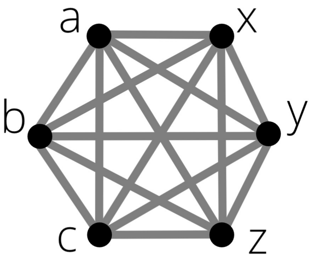

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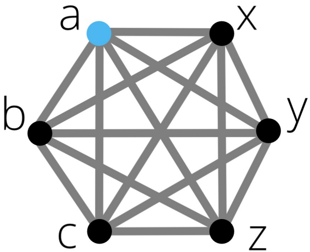

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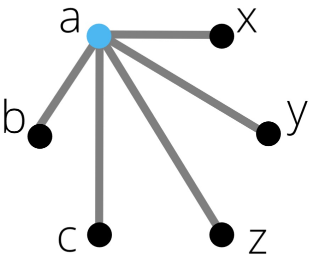

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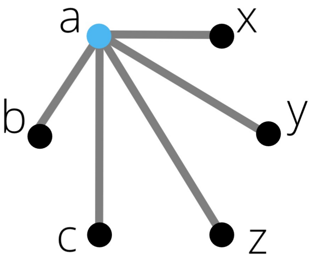

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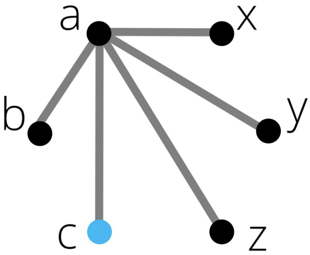

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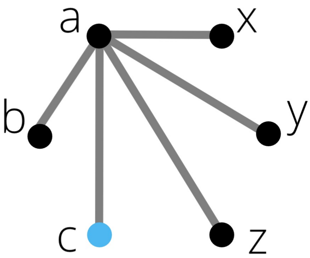

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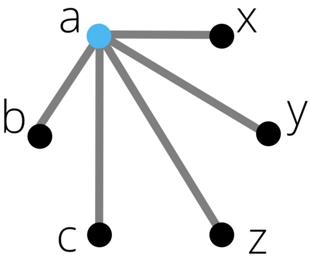

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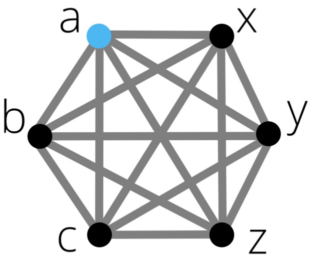

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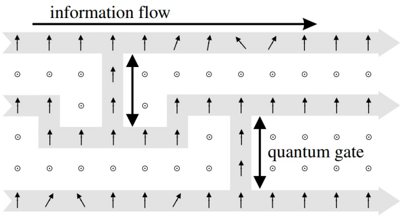

FIG. 1. Quantum computation by measuring two-state parti-

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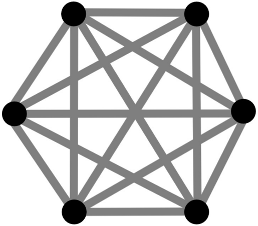
Want a structure that guarantees some H' is not a vertex-minor.

Difficulty: Our graph class can have arbitrarily large cliques.



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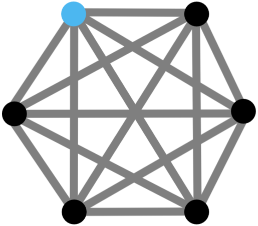

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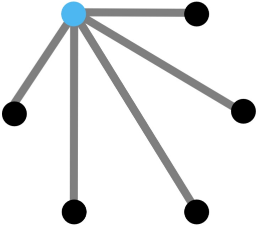

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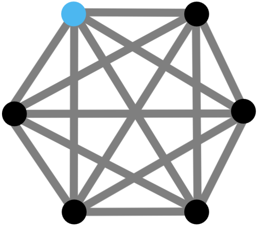

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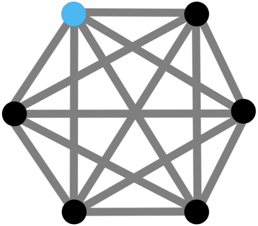
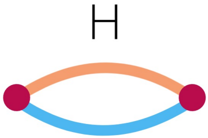
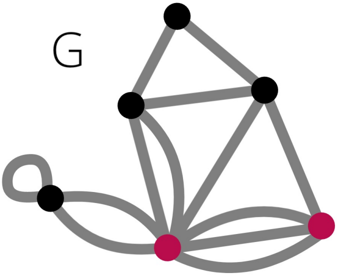


Approach: Kotzig and Bouchet found a connection with **flooding immersions**.

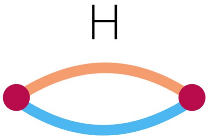


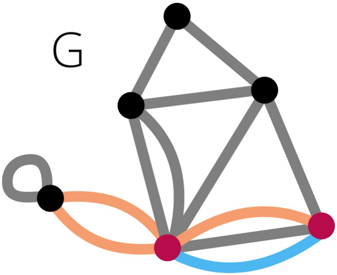


An **immersion** of H into G consists of

- an injection $\psi : V(H) \to V(G)$ and
- for each $e = uv \in E(H)$, a $(\psi(u), \psi(v))$-trail in G, s.t. the trails are edge-disjoint.

It is **flooding** if every edge of G is in one of the trails; we say H **floods** G. If H and G are Eulerian, every immersion can be made flooding.





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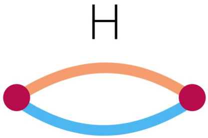

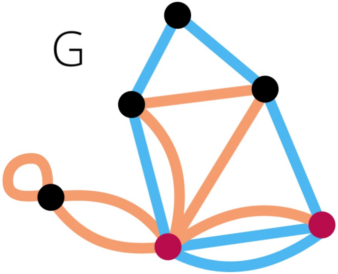

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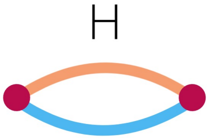

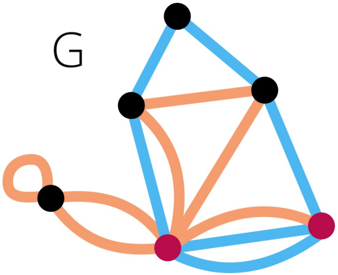

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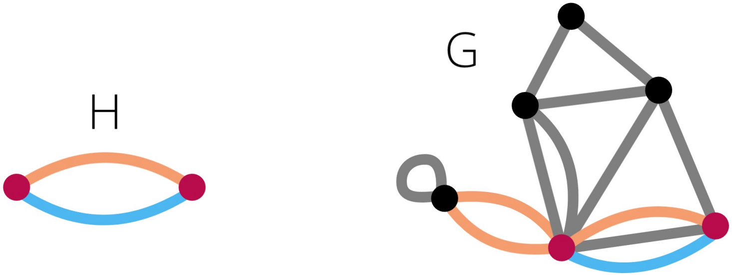

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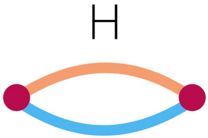

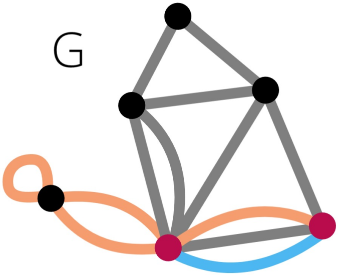

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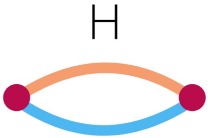

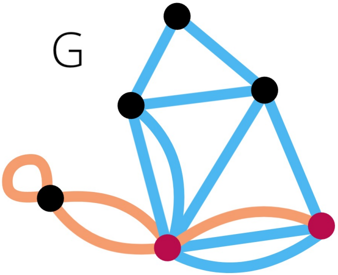

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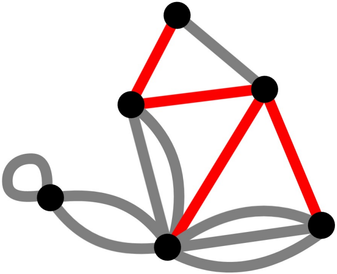
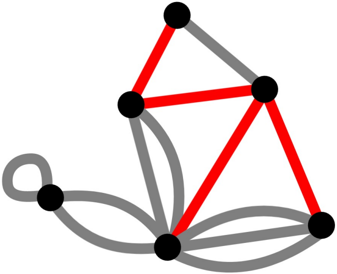
- an injection $\psi : V(H) \rightarrow V(G)$ and
- for each $e = uv \in E(H)$, a $(\psi(u), \psi(v))$-trail in G, s.t. the trails are edge-disjoint.

It is **flooding** if every edge of G is in one of the trails; we say H **floods** G. If H and G are Eulerian, every immersion can be made flooding.



A **signed graph** is a graph with a specified set Σ of edges, called the **signature**. Equivalently, each edge e has a **weight** $w(e) \in \mathbb{Z}_2$.

We only care about the **weight** of each cycle, where $w(C) := \sum_{e \in E(C)} w(e)$ over $\mathbb{Z}_2$. So **re-signing** (adding 1 to each edge in a cut) gives an equivalent signed graph.



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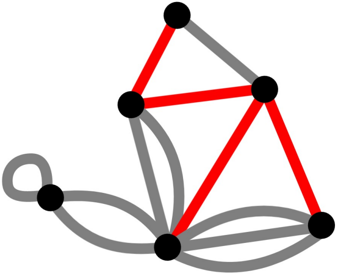

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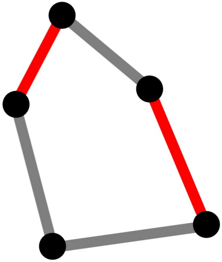

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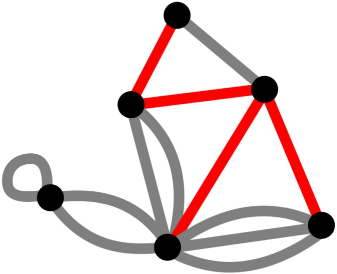

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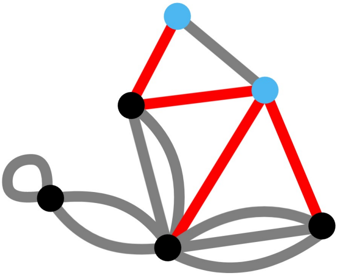

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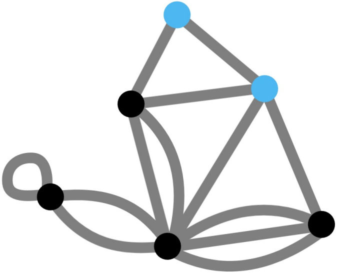

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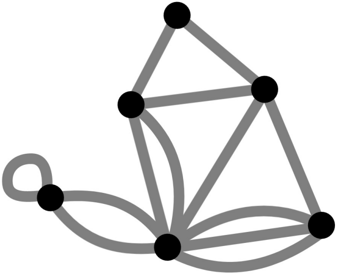

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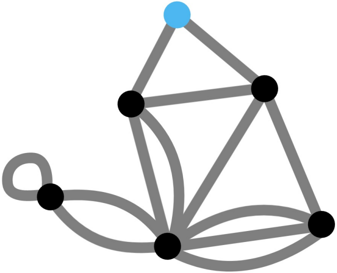

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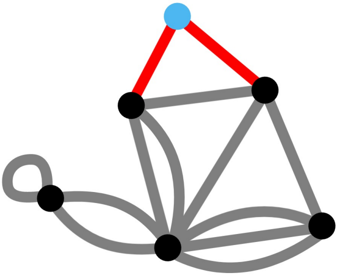

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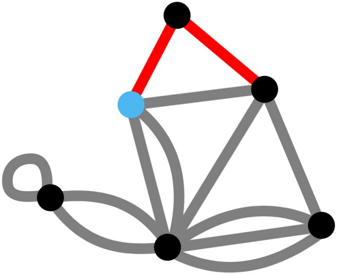

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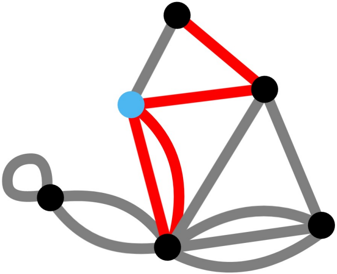

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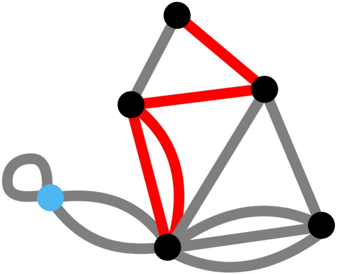

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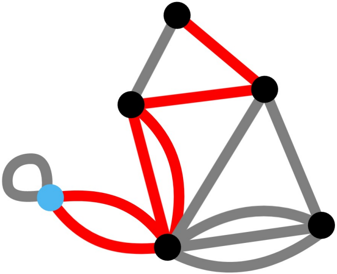

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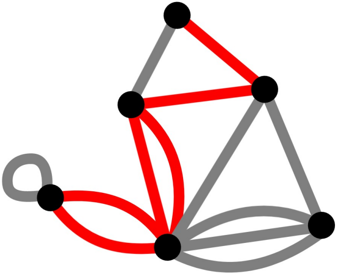

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- each $e \in E(H)$ is sent to a trail of weight $w(e)$ in G.

Now **flooding immersions** behave much differently.

Graphs with k different signatures are called **$\mathbb{Z}_2^k$-labelled**; each edge has a weight $w(e) \in \mathbb{Z}_2^k$. We can **re-sign** on any $\gamma \in \mathbb{Z}_2^k$.

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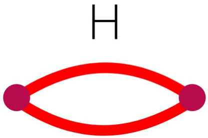

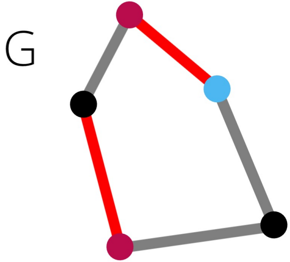

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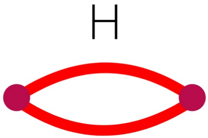

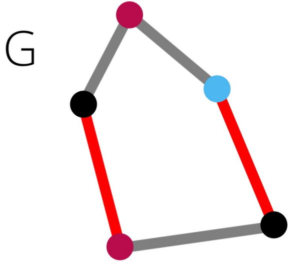

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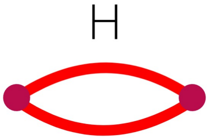

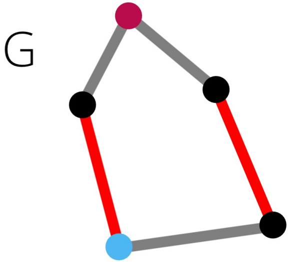

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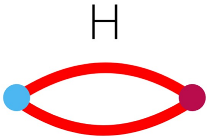

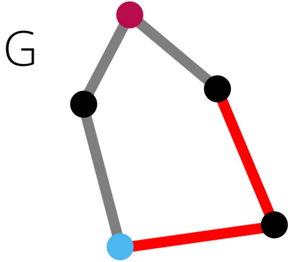

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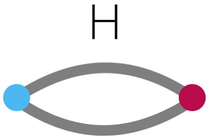

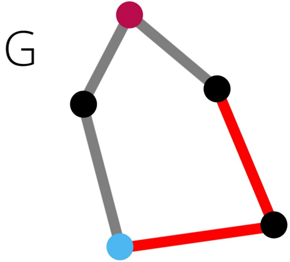

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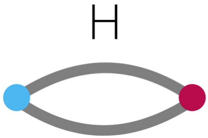

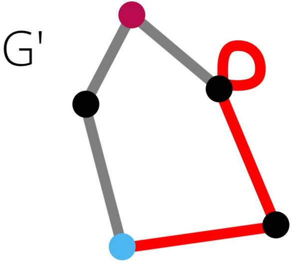

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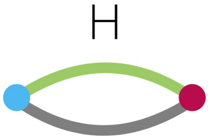

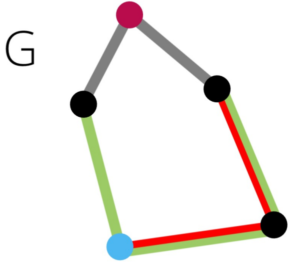

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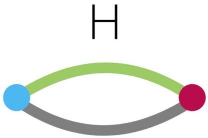

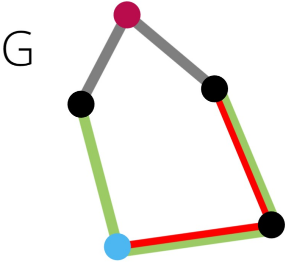

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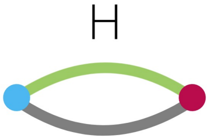

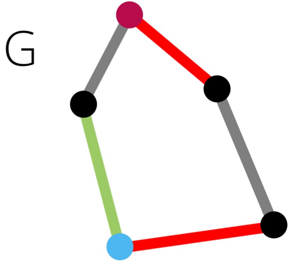

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The connection is through **circle graphs**.

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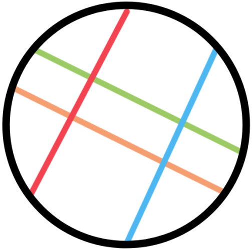
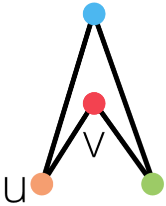


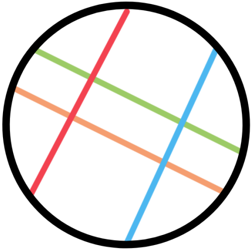
chord diagram



circle graph G

A **circle graph** is the intersection graph of chords on a circle.
Circle graphs are closed under local complementation.

View the **chord diagram** as a 3-regular graph and contract the chords to get the **tour graph**. It is invariant under local complementation. Here is how we delete v.



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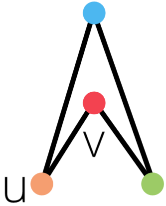

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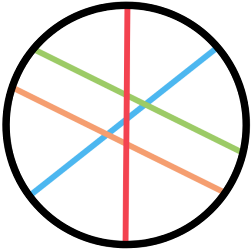

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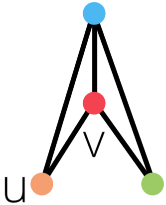

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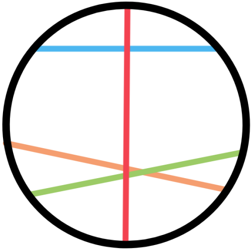

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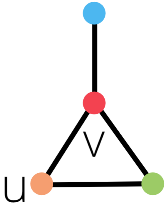

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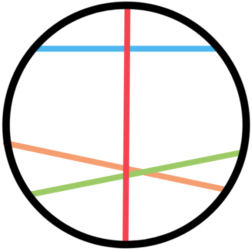

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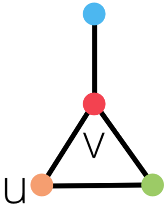

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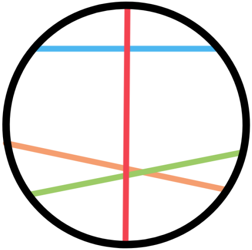

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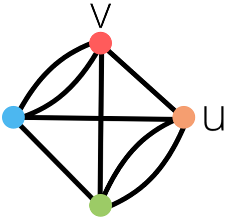

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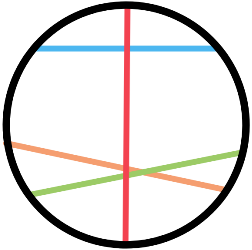

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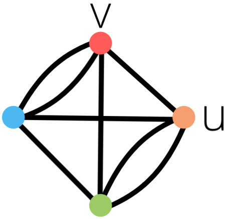

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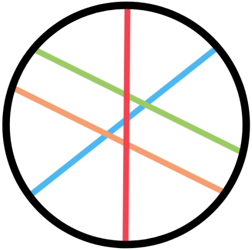

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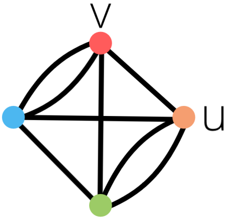

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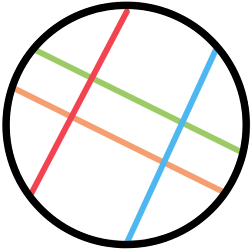

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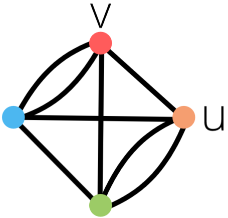

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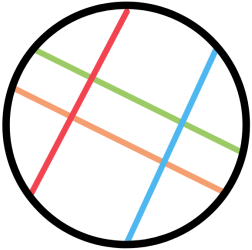

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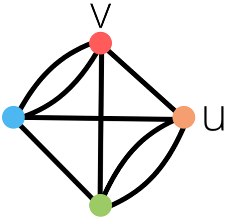

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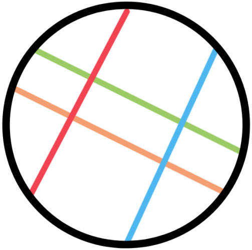

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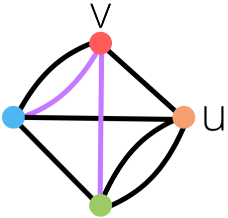

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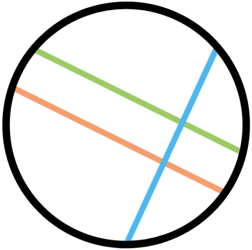

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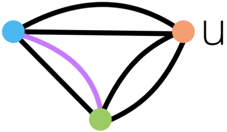

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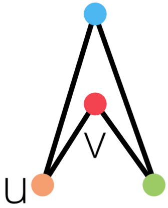

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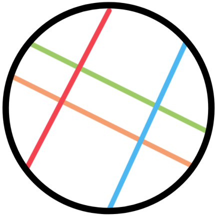

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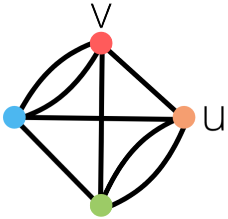

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Theorem (Kotzig 77 and Bouchet 94)

If H and G are prime circle graphs, then

- *their tour graphs $T(H)$ and $T(G)$ are unique and*
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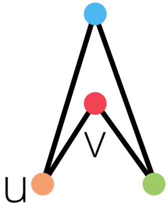

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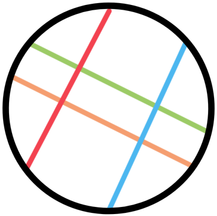

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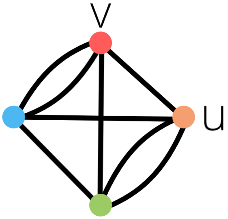

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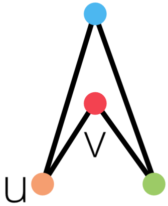

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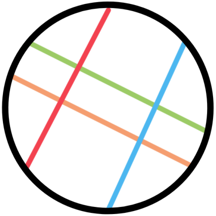

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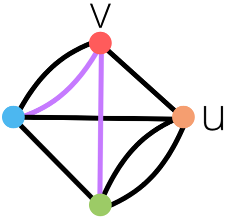

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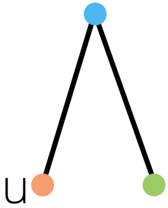

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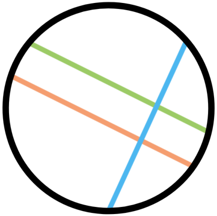

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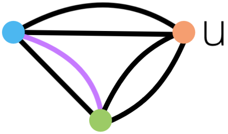

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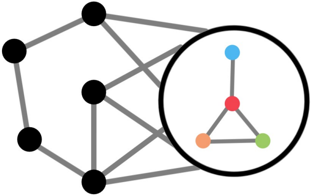

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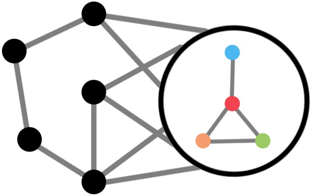

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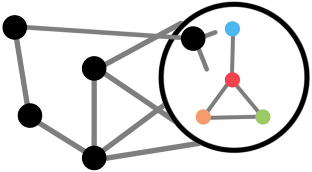

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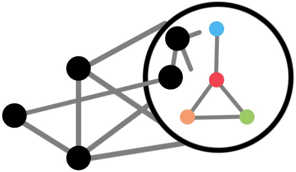

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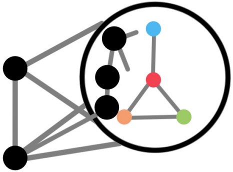

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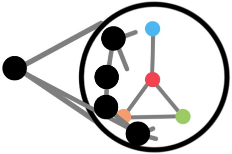

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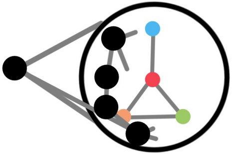


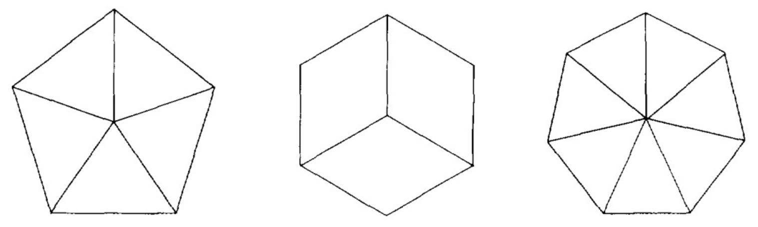
WMA that after local complementation, our favorite circle graph is an induced subgraph of G. Throw vertices into the circle graph. The remaining vertex gives a **signature** on the **tour graph**.

Theorem (Bouchet 94)

A graph is a circle graph

$$\iff$$

it has none of



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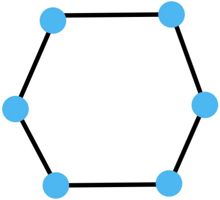
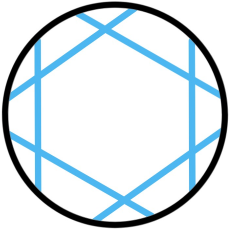
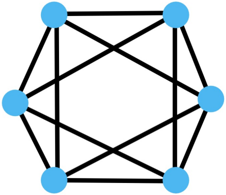


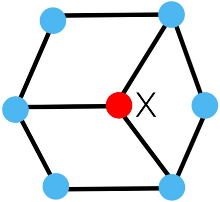
circle graph



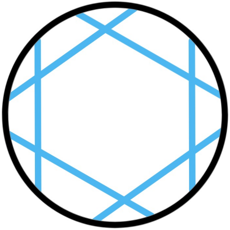
chord diagram



tour graph



circle graph + x



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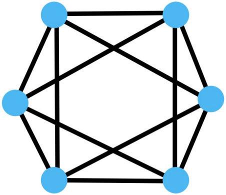

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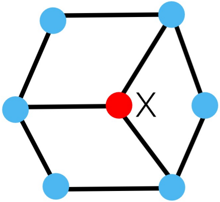

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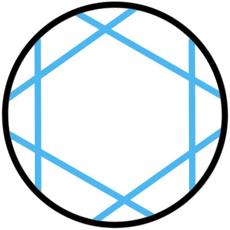

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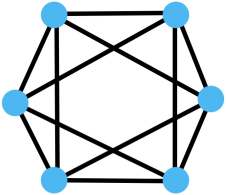

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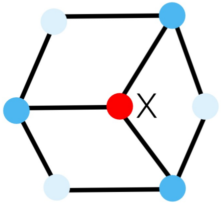

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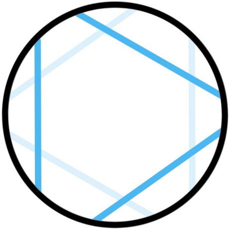

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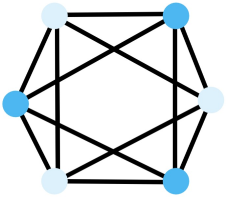

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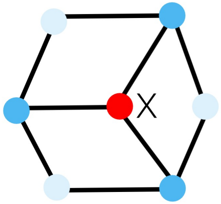

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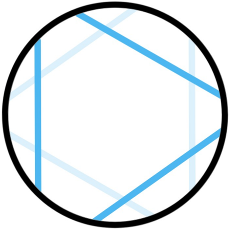

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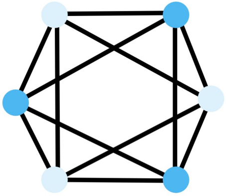

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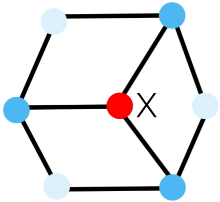
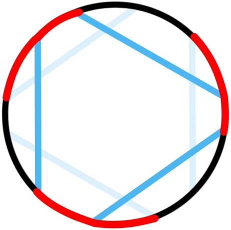
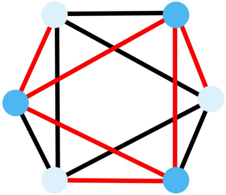


circle graph + x



chord diagram



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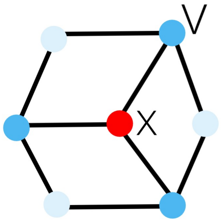

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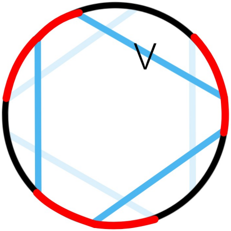

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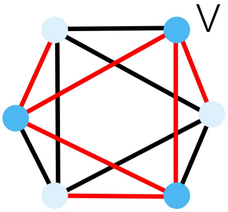

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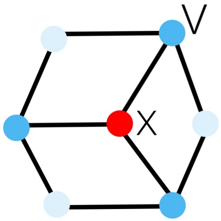

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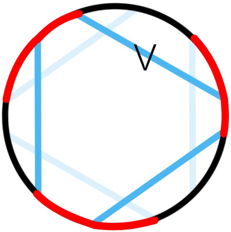

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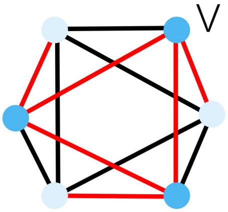

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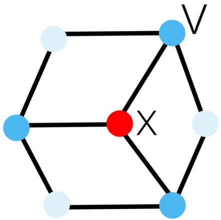

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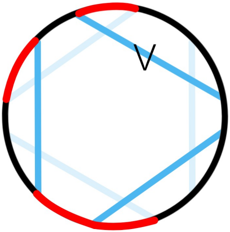

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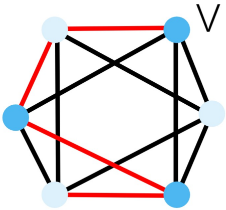

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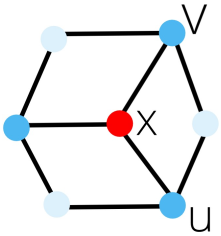

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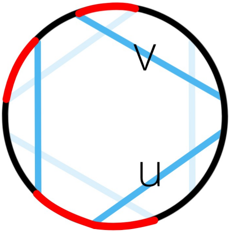

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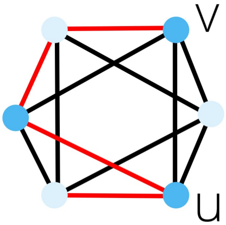

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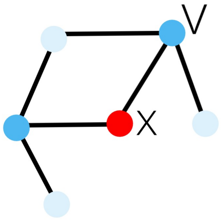

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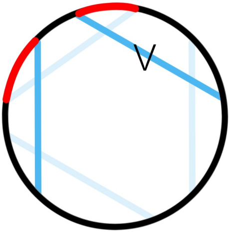

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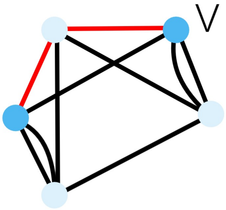

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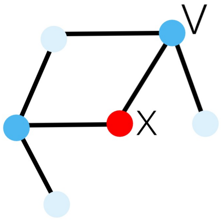

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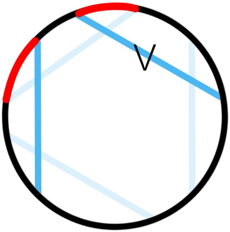

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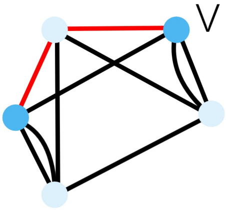

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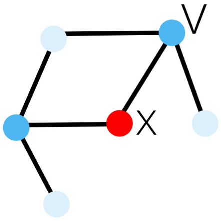

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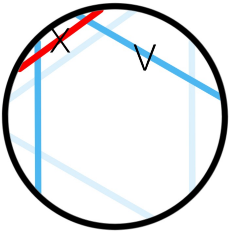

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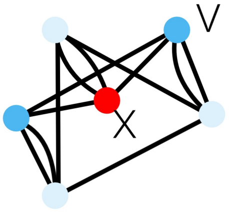

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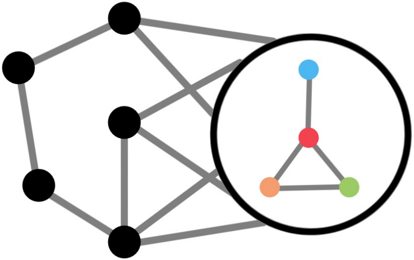

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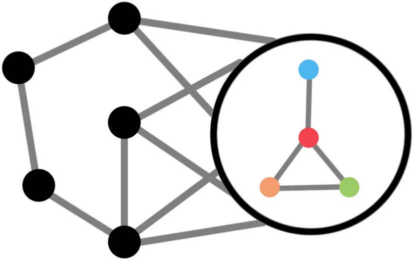

Then the **tour graph** is $\mathbb{Z}_2^k$-**labelled**.

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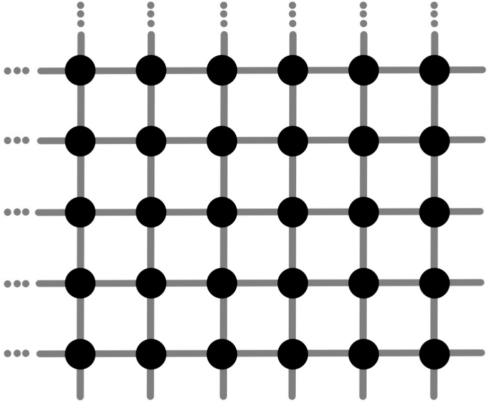

A toroidal grid with signatures Σ_1, Σ_2, Σ_3 of size 4, "far apart".

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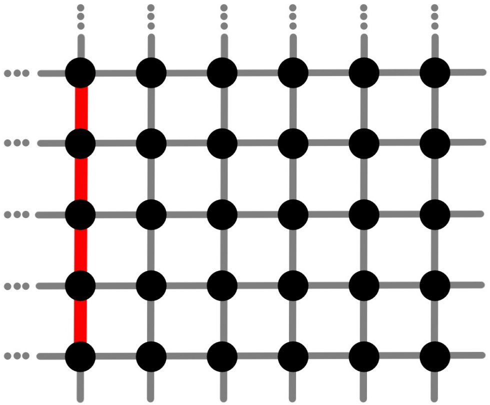

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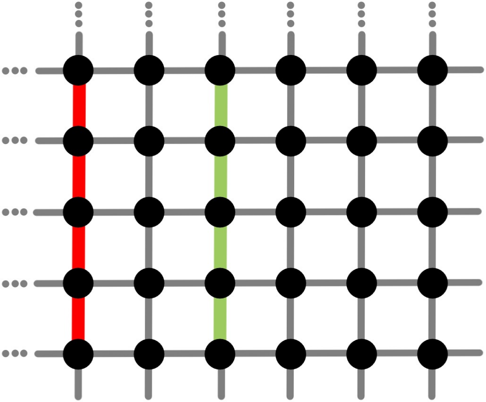

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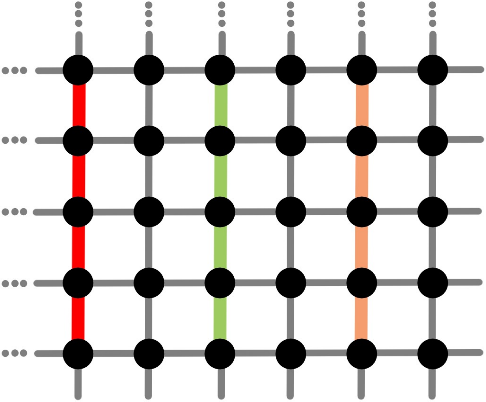

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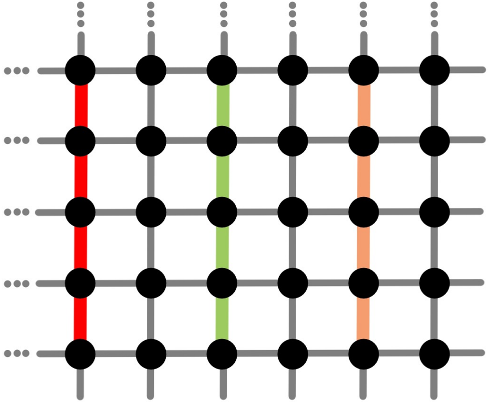

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Suppose we have a grid subgraph

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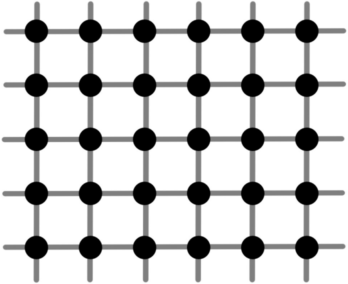

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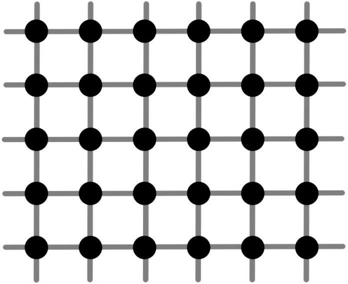

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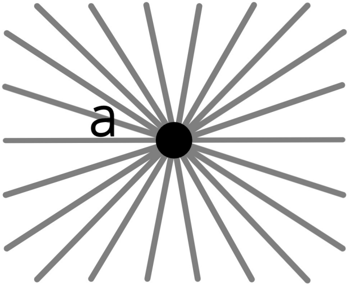

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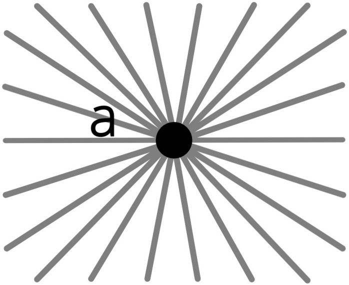

Need a "non-zero *A*-paths" type result...

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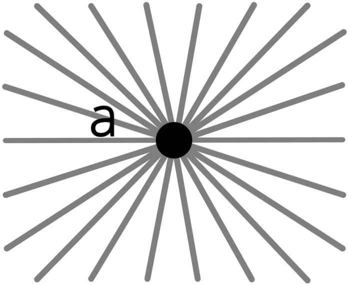

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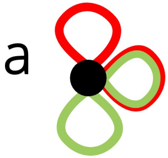

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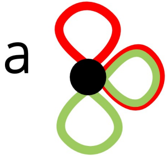

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Have precise min-max theorem for $\mathbb{Z}_2^k$-**labelled** graphs.

Corollary

If G is $\mathbb{Z}_2^k$-labelled, Eulerian, and $2d$-**edge-connected** *for an integer $d \geq 2$, and $a \in V(G)$ with* $\max t < d$, *then there exist $S \subseteq V(G) \setminus \{a\}$, $\gamma \in \mathbb{Z}_2^k$, and a re-signing s.t.*

1. *every non-zero edge is incident to a vertex in S and has weight γ, and*
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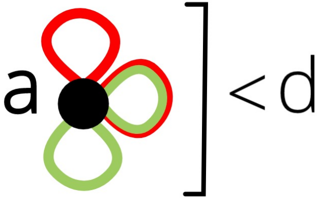

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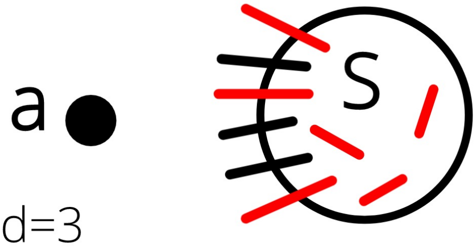

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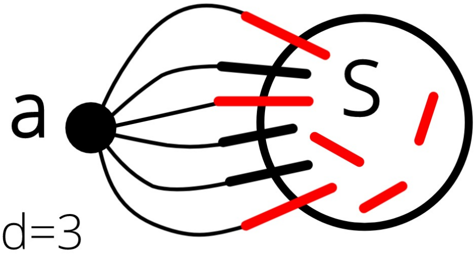

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Conjecture: Graphs are well-quasi-ordered by vertex-minors.

- See (Oum 08).

Thank you!